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Transaction costs and crowding

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We use industry data to determine whether crowding of the investment space is caused by portfolio construction processes typical to the investment community. In particular, this paper examines the extent that transaction cost models cause crowding of the investment space, even when the investment models are completely unrelated to one another. We find that as transaction costs become more significant in the portfolio creation process as portfolios increase in size from $500 million to $5 billion, crowding actually declines for long-only portfolios and mainly declines, but sometimes increases for market neutral portfolios. This research sheds more light on how crowding develops through actions by players within the financial system.

Keywords: Risk management; Crowding; Crowded spaces; Transaction costs; Copycat trading; Quantitative equity portfolio management; Optimal portfolios; Portfolio construction

JEL Classification: G0, G01, G02, G11

1. Introduction

Although financial crises have various origins, they are frequently caused or at least amplified by trade crowding in the investment space (Chincarini, 1998, 2012). Crowding can take place through a variety of mechanisms. First, trading spaces may become crowded if investors follow similar trading models, either by coincidence or intentionally, because this makes it likely that their resulting portfolios will be very similar. The key feature that makes the space crowded is that the number of portfolio managers chasing a similar strategy is too large given the available liquidity or typical turnover. A series of studies has used institutional holding data to document how copycat trading can lead to crowding (Choi 2013, Kim and Zhang 2013, Spilker 2016).

Crowding can also occur when investors use similar techniques to construct their portfolios. Even if they have different models for generating their expected returns, investors' use of similar techniques for portfolio construction can cause their portfolios to converge. Some studies using institutional holding data from the United States and Sweden have shown that similar portfolio structures can lead to crowding (Bohlin and Rosvall 2014, Balagyozyan and Cakan 2016). One component of portfolio construction considers the explicit and implicit costs of trading securities. If a group of similar portfolio managers have similar transaction cost models, their portfolios might be more similar than they would be in the absence of transaction costs.

Crowding can be a problem for investors because it alters the risk and return dynamics of a trade (Pojarliev and Levich 2011, Cahan and Luo 2013, Yan 2013, Ibbotson and Idzorek 2014, Menkveld 2014). Specifically, it makes the risk of a trade endogenous to the trade itself. Some research on institutional holding data shows that crowded hedge fund or institutional holdings can lead to distorted risks and returns (Blocher 2013, Lou and Polk 2013, Anton and Polk 2014, Bayraktar *et al*. 2015), and one analysis of the stock ownership of institutional investors shows that higher crowdedness in stock ownership leads to substantially higher liquidity risk (Beber *et al*. 2014). However, other research suggests that hedge fund crowding does not distort markets (Reca *et al*. 2014).

Some hedge fund managers and quants maintained that the quant crisis of 2007 was caused by crowding.† Some argued that it was crowding of the alpha models (Cerla 2007, Chincarini 2012, Khandani and Lo 2007, Rothman 2007), but others argued that it was really about liquidity and that transaction cost models may have crowded the types of trades that were made (Chincarini 2012)—that is, market impact costs might have led quant funds with large portfolios to trade only a

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†The quant crisis refers to an event that occurred during the period 2 August to 8 August 2007. During this period, a group of highly successful quantitative long/short equity funds that used quantitative equity strategies suffered extreme losses in their portfolios; these firms included AQR, Blackrock, Goldman Sachs, State Street, and many others. It is commonly believed that the extreme losses occurred due the firms' crowding of investment positions, which led to a fire sale liquidation of similar portfolios that happened to be quantitatively constructed. For a firsthand account of the crisis, see chapter 8 of Chincarini (2012).

handful of very liquid securities, which caused extreme movements in these securities. For example, according to Mark Carhart, former Co-CIO of Quantitative Strategies at GSAM and Founder of Kepos Capital, 'Crowding among quants happens for several reasons, but the transaction costs model was of primary importance, as it caused us to trade similar securities at each point in time' (Chincarini 2012).

This paper studies the interacting roles of transaction cost models, portfolio construction, and crowding, and makes several contributions to the literature on crowding and liquidity. First, it helps to clarify how transaction cost models contribute to crowding in the investment space. Second, it introduces a simple method for approximating several varieties of transaction costs that can be used in portfolio optimization. The approximation is very accurate and quite simple to use, and practitioners can use it to model a variety of complex transaction costs within a standard portfolio optimization framework.

Additionally, this paper presents a simulation of equity portfolio managers that act independently of one another; the simulation shows that as the average portfolio size grows from \$500 million to \$5 billion in assets, transaction costs do not increase crowding. In fact, crowding actually decreases in this range. As the average portfolio grows from \$5 billion to \$20 billion, crowding begins to increase; however, it is still not significantly different from a scenario involving portfolios that do not consider transaction costs. Thus, the evidence in this paper suggests that when transaction costs are properly integrated in the portfolio management process, they are unlikely to cause crowding problems for reasonably sized portfolios. For example, it is more likely that the crowdedness that made it difficult for portfolio managers to exit positions in August 2007, was caused by crowded alpha models or less disciplined consideration of transaction costs in the portfolio building process.

The rest of this paper is organized as follows. Section 2 defines the measure of crowding that will be used in the paper. Section 3 presents our empirical framework for examining the crowding that arises from portfolio construction and transaction costs. Section 4 describes the transaction cost models we use to construct portfolios for the simulation. Section 5 discusses the empirical simulation procedure and the results from the simulated portfolios. Finally, section 6 concludes the paper.

2. A definition of crowding

For the purposes of this paper, we define crowding to be when investors own portfolios with similar holdings. Let the similarity between two portfolios be measured by s_{ij}, which is the dot product between the position weight vectors ($\mathbf{w}$) of each portfolio i and j divided by the product of the Euclidean norm of each vector. Thus,

$$s_{ij} = \frac{\mathbf{w}_i' \mathbf{w}_j}{|\mathbf{w}_i||\mathbf{w}_j|} \tag{1}$$

where the Euclidean norm is defined across N assets as

$$|\mathbf{w}_i| = \sqrt{\sum_{n=1}^{N} \mathbf{w}_{in}^2} \tag{2}$$

This measure will have a value between 0 and 1 for portfolios that can only be long securities (i.e. long-only portfolios). This measure will have a value between −1 and 1 for portfolios that can have negative weights.†

In our paper, we will study more than just two portfolios. Thus, for studying a group of M portfolios, we define the N-by-M portfolio holdings matrix as the matrix, H, which consists of columns of position weight vectors on N assets for each of M portfolios. The similarity matrix amongst all portfolios is computed as

$$S = \left(H'H\right) \circ \hat{\hat{H}} \tag{3}$$

where $\circ$ represents the Hadamard product or the element-by-element multiplication of the matrices, and

$$\hat{\hat{H}} = \begin{bmatrix} \frac{1}{\hat{h}_{11}} & \frac{1}{\hat{h}_{12}} & \cdots & \frac{1}{\hat{h}_{1M}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\hat{h}_{M1}} & \frac{1}{\hat{h}_{M2}} & \cdots & \frac{1}{\hat{h}_{MM}} \end{bmatrix} \tag{4}$$

and $\hat{H} = |H|'|H|$, where $|H|$ contains the Euclidean norm of each manager's weight vector. The matrix S contains the similarities of each portfolio with every other portfolio. For example, element S_{12} represents the similarity of the portfolios of managers 1 and 2. For a specific set of portfolios, our measure of crowding is given by the average of the off-diagonal elements of this matrix.‡

From the similarity matrix of M portfolios or portfolio managers, we measure the crowding, C, amongst the group of portfolios as the average similarity between portfolios.§

$$C = \frac{\sum_{i=1}^{M}\sum_{j=1}^{M} S_{i,j} - M}{M^2 - M} \tag{5}$$

A simple example with a universe of three portfolios holding three stocks each might help to illustrate the concept of crowding. Suppose our matrix H of manager holdings is given as

$$H = \begin{bmatrix} 0.4 & 0.8 & 0.45 \\ 0.4 & 0.1 & 0.45 \\ 0.2 & 0.1 & 0.10 \end{bmatrix}. \tag{6}$$

This example includes the portfolios of 3 managers. Each manager has a portfolio whose holdings sum to 1. The portfolio of manager 1 has 40% in stock 1, 40% in stock 2, and 20% in stock 3. Portfolio 2 has 80% in stock 1 and 10% in stocks 2 and 3. Portfolio 3 has 45% in stock 1 and 2 and 10% in stock 3.

†This measure is related to more commonly used measure known as Pearson correlation. One can think of Pearson correlation as a demeaned version of cosine similarity.

‡The diagonal elements are the similarity of each portfolio with itself, which are irrelevant. Our measure of the similarity of portfolios to measure crowding is related to a more commonly known cosine similarity, which is a measure of the similarity between two vectors of an inner product space that measures the cosine of the angle between them. This measure is given as $\theta = \cos^{-1}\left(\frac{\mathbf{w}_i'\mathbf{w}_j}{\|\mathbf{w}'_i\|\|\mathbf{w}_j\|}\right)$.

§The first term of the numerator represents the summation of all the similarities between every portfolio manager and every other, including it's own. By subtracting M, we normalize this measure to be the average similarity in excess of a group of portfolio managers that are completely dissimilar to each other. In that case, the similarity matrix would be a diagonal of 1s.

One can see that manager 1 and manager 3 have very similar or 'crowded' portfolios. Manager 2's portfolio is less related to the other two. Using our formula for computing the similarity matrix, we find that†

$$S = \begin{bmatrix} 1 & 0.7796 & 0.9831 \\ . & 1 & 0.7906 \\ . & . & 1 \end{bmatrix}. \qquad (7)$$

The resulting similarity matrix corresponds with our intuition. That is, portfolios 1 and 2 have a similarity measure of 0.7796, which is high, but not as high as portfolios 1 and 3, which have a measure of 0.9831. For this universe of portfolios, the crowding measure is $C = 0.8454$. This indicates that there is a high level of similarity or crowding in the investment space from these three portfolio managers.

In addition to capturing the crowding of a group of portfolios or portfolio manager holdings, we also wish to study specifically how the portfolio construction process creates additional crowding in the investment space. One way to do this is to measure the crowding of portfolios *before* the portfolio construction process and *after* the portfolio construction process. Specifically, if we were able to observe the expected return models or alpha models of portfolio managers before they assigned weights to their portfolios, we could infer the amount of crowding that is added or removed from portfolio construction techniques.

Let's define S_α as the similarity matrix of portfolio managers from their alpha models. This is the similarity of their stock picking models, whether quantitative or qualitative managers. Define S_p as the similarity of portfolios after the manager has combined his alpha model with his optimization model to construct his final portfolio. Thus, S_p is the similarity matrix of actual portfolio holdings. Both measures are computed as described previously. For both S_α and S_p, the crowding measures are also computed and given by C_α and C_p. In our analysis, we will compute the ratio of these two as

$$\Omega = \frac{C_p}{C_\alpha}. \qquad (8)$$

Omega (Ω) measures the ratio of the crowding of actual portfolios after the optimization process which explicitly considers transaction costs to the crowding of the portfolio alpha signals. When this ratio is greater than one, it means that the portfolio construction process has caused portfolios to become more crowded than they were just from the different portfolio manager beliefs about the attractiveness of different stocks and vice versa. In other words, this metric represents how much more similar on average the portfolio of holdings are than the expected return models.‡

In our simulation analysis, we will look at the crowding of portfolios, C, as well as the ratio of crowding before and after portfolio construction, Ω.

3. The empirical framework

Our strategy to analyse the real-world implications of crowding from the portfolio construction process is to simulate the construction of portfolios using random alpha signals, as well as factor model alphas, combined with real-world risk models and realistic transaction cost models in order to examine the extent of crowding that occurs indirectly due to the transaction cost considerations of portfolio managers.§ The data for our empirical study was obtained from several sources. We obtained monthly stock returns and stock fundamental data from Factset. We obtained monthly risk model parameters from the three major risk model providers in the financial industry; Barra, Axioma, and Northfield. For each risk model, we obtained all components of the risk factor models on a monthly basis so we could create the monthly variance–covariance matrix for all stocks in the universe. The sample covers the period from 2006 to 2013 using monthly stock data of the largest 2000 stocks in the United States based on their market capitalization at the end of the month.

In this section, we discuss the techniques of our simulation process, including the creation of our expected return or α models for stocks, the risk models, and the portfolio construction techniques.

3.1. Portfolio construction

In order to examine the extent of crowding from the portfolio construction process due to transaction cost considerations, we created portfolios that were more common in the professional investment world. We considered two types of portfolio management techniques; a long-only portfolio and a market neutral portfolio.¶

†The components of S are given by,

$$H'H = \begin{bmatrix} 0.36 & 0.38 & 0.38 \\ . & 0.66 & 0.415 \\ . & . & 0.415 \end{bmatrix},$$

$$\hat{H} = \begin{bmatrix} 0.36 & 0.4874 & 0.3865 \\ . & 0.66 & 0.5234 \\ . & . & 0.4150 \end{bmatrix},$$

and

$$\hat{\hat{H}} = \begin{bmatrix} 2.778 & 2.0515 & 2.5872 \\ . & 1.5152 & 1.9108 \\ . & . & 2.4096 \end{bmatrix}.$$

‡We could have also taken the average of the absolute values in this similarity matrix. This would not be as representative of crowding itself, but would also be important for a measure of financial fragility. That is if 50% of managers are long a portfolio and 50% of managers are short a portfolio, our current measure would have a lower level of crowding than the absolute measure. However, this particularly extreme case might indicate a very fragile financial system when crowding is considered in this broader context. One could also consider measuring the absolute value of stock weights when computing the similarity matrix, however, this would not represent crowding as much as it would represent activity in similar stocks.

§The procedures used are very similar to those used by sophisticated portfolio managers. For example, Goldman Sachs quant equity group managed portfolios in a similar way. 'Our approach to portfolio construction uses these individual company alphas in combination with other optimization criteria with the goal of maximizing each portfolio's risk-adjusted expected return net of transaction costs. The inputs to our optimization process are return forecasts, transaction cost estimates, risk estimates, and of course, client objectives. Our risk model and risk forecasts are central to the optimization process.' (Daniel 2008).

¶For more details on the optimization process, see appendices 1 and 2.

The long-only portfolio manager maximizes net expected return (i.e. returns after transaction costs) whilst keeping a portfolio that has a volatility equal to the historical volatility of the S&P 500, is not levered, has a maximum stock weight of 10% in any one name, and whose sector composition matches that of the benchmark.† We also consider the same long-only portfolio manager, however, rather than maximize alpha subject to a risk target, the portfolio manager minimized risk subject to a net expected return target.‡

The market neutral manager maximizes expected return subject to having no more than 5% volatility over the risk-free rate, a dollar-neutral portfolio (i.e. the weights of the longs sum to the weights of the shorts), a leverage of 2 (i.e. the weights of the long portfolio sum to 1 and the weights of the short portfolio sum to 1), that no stock can have a weight less than −10% or more than 10%, and that the stocks are sector neutral with respect to the long and short side of the portfolio.§ We also considered a market neutral manager that minimized the volatility of the portfolio subject to a target alpha.¶

For both types of portfolio construction, we also considered liquidity constraints, that is we constrained the manager to not purchase too much of a certain stock with respect to the average daily trading volume, but did not report these results in the paper.‖

3.2. The alpha models

3.2.1. Random alphas. In order to focus on the amount of crowding that is caused from the portfolio construction process when considering transaction costs, we used random alpha models for the different portfolios. That is, each portfolio manager receives signals about the stock universe that are random. Thus, the degree of crowding from the alpha models, prior to portfolio construction, has an average value of zero.

In order to construct the random alpha signals for each portfolio manager, we drew 100 random alpha signals for all stocks from a normal distribution, $\boldsymbol{\alpha} \sim N(0, \boldsymbol{\Sigma_\alpha})$, where $\boldsymbol{\Sigma}$ is the historical variance–covariance of asset returns up to the time of portfolio selection with the off-diagonals set to 0.†† That is, we used the historical volatility of each asset, but ignored the correlations.‡‡ These random alpha signals can also be thought of as that the expected returns that each manager generates for each stock prior to building his or her portfolio which are randomly selected from the distribution of returns for that particular stock.

3.2.2. Factor-model alphas. Although the random models provide the greatest insight into the crowding from the portfolio construction process, we also examined how crowding occurs when expected return or alpha models are more realistic. Thus, we constructed a series of alpha signals that were built from three commonly accepted factor models for stock returns.§§ The first factor is the value factor (Chincarini and Kim 2006, Fama and French 1992, Lakonishok *et al*. 1994). Our value factor is constructed as the most recently reported book value per share divided by the price of the security at the time of portfolio formation. The second factor is the momentum factor (Jegadeesh and Titman 1993, Chan *et al*. 1996, Carhart 1997, Grinblatt and Moskowitz 2004). Our momentum factor is the 11 month compounded return for each security lagged 1 month from the date of portfolio formation. The third factor is the beta factor. This takes advantage of the beta anomaly; that is, that high betas tend to underperform and low betas tend to outperform (Black *et al*. 1972, Blume 1975, Fama and French 1992, Jagannathan and Wang 1996, Chincarini *et al*. 2013, Frazzini and Pedersen 2014). Our beta factor is the historical beta for each stock using the last 60 months of returns for the security and the S&P 500 total return as the market return and running an OLS regression of stock returns against the market return. We take the negative of this value.

For these three factors, we compute a Z-score using the standard formula,¶¶

$$z_{it} = \left(\frac{f_{it} - \bar{f}_t}{\sigma_{f_t}} \right) \tag{9}$$

where f_i is the factor value for stock i, $\bar{f}_t$ is cross-sectional mean across stocks at time t, and σ_{f_t} is the cross-sectional standard deviation at time t.

†These portfolio parameters are quite reasonable. In fact, we surveyed several portfolio managers before creating our parameters. We also experimented with other maximum and minimum weights for the portfolio. The benchmark portfolio for the purposes of sector neutralization was the top 2000 companies selected by market capitalization each month and weighted by market capitalization.

‡The net expected return target was chosen to be the historical annualized volatility of the S&P 500 divided by $\sqrt{12}$. There was no specific reason for these choices, except that they seemed to be reasonable. If a specific target could not be achieved, we searched for the next reasonable target.

§These portfolio parameters are quite reasonable. In fact, we surveyed several portfolio managers before creating our parameters. While it is true that different managers may use slightly different parameters, the main purpose of this paper is to describe the potential crowding effects that may occur when portfolio managers use reasonable parameters and similar risk models.

¶The target alpha was the same as with the long-only case.

‖We did not include liquidity constraints for the randomly generated alphas, because for large portfolios, liquidity constraints could not be kept at the 30% level of average daily trading volume (ADV). They had to be increased up to 70% for the portfolio optimizer to solve. For market neutral portfolios with 2x leverage it was even more difficult to satisfy these constraints. This has interesting implications for portfolio management. As a portfolio increases in size and one seriously considers liquidity constraints, the portfolio manager must either accept to trade over several days and accept an increasing position over time or the portfolio manager must increase the portfolio tolerance as a function of average daily trading volume. Both of these increase the problems with liquidity and crowding in an exit situation.

††The reason for choosing 100 random draws rather than a larger number had to with the tradeoff between sufficiently large and the computation time required. To create the 100 random portfolios for 12 months of data took 20 days on a supercomputer that used 12 cores.

‡‡Further research might wish to consider a random model which draws from a standard normal distribution, $\boldsymbol{\alpha} \sim N(0, 1)$, where signals for individual assets are independent of their historical volatility. Further research may also wish to consider a model that draws from a full variance–covariance matrix of asset returns rather than just the diagonals.

§§All of our realistic alpha models were constructed from fundamental stock data obtained from Factset's database. All data were lagged so as to avoid look-ahead bias.

¶¶For more info, see Chincarini and Kim (2006).

We use the Z-scores to represent the alphas for the assets. Thus, the higher the Z-score for a stock in a given month t, the more attractive that stock for our portfolio.

Before constructing the portfolios, two further adjustments to the Z-scores were needed. First, outlier data for individual factors were removed using the interquartile range procedure. That is, we computed the third quartile entry of every factor (Q3) and the 1st quartile entry of every factor (Q1). Then we computed the interquartile range (IQR) as Q3–Q1. We then computed an upper and lower bound for the factor as,

$$UB = Q3 + 3IQR \tag{10}$$

$$LB = Q1 - 3IQR \tag{11}$$

We then labelled all stocks with factor values above the upper bound and below the lower bound to be outliers.† For these, we set their values to missing and computed the Z-scores for the remaining stocks. For the outlier stocks, we fixed the Z-scores at the maximum and minimum of the non-outlier stocks' Z-scores.‡

Second, in order to use the Z-scores as expected return or alpha signals and as after-tax expected returns, we converted them to units of return. To do this, we chose to keep the relative Z-score values between stocks, but convert the distribution to resemble the first two central moments of the cross-section of stock returns. Thus, the Z-scores for each stock were modified as follows:

$$\tilde{Z}_{it} = Z_{it}\sigma_{rt} + \mu_{rt} \tag{12}$$

where $\tilde{Z}_{it}$ is the adjusted Z-score to be used as the alpha for each stock, σ_{rt} and μ_{rt} are the cross-sectional standard deviation and mean of stock returns over the last twelve months divided by 12 at time t.

Given this conversion of Z-scores towards the cross-sectional stock distribution of returns, we then compute the after transaction costs for a given stock as:

$$\tilde{\alpha}_{it} = \tilde{Z}_{it}w_{it} - \widehat{tc}_{it} = \tilde{Z}_{it}w_{it} - \hat{a}_{it}w_{it} - \hat{b}_{it}w_{it}^2 \tag{13}$$

where $\tilde{\alpha}_{it}$ is the after-transaction cost alpha for stock i at time t, w_{it} is the weight in stock i at time t, and $\widehat{tc}_{it}$ are the approximate transaction costs from the regression equations.§

An example of this conversion of Z-scores is shown in figure 1 for the December 2013. The first graph in figure 1 shows the distribution of the Z-scores amongst 2000 stocks according to a particular Z-score model. The second graph shows the distribution of stock returns for the same stocks. The third graph shows the distribution of the modified Z-scores adjusted for the mean and standard deviation of the cross-section of stock returns. In this particular example, the mean and standard deviation of the cross-section of stock returns are 4.22% and 4.71%. The mean and standard deviation of the Z-scores are 0 and 1 and the modified Z-scores have the same mean and standard deviation as the stock returns.

†Other values for the upper and lower bound could be chosen by using 1.5 instead of 3. It is common to use both.

‡This procedure works quite well at dealing with outlier data. For example, in December 2013, the raw data for the 5-year beta of company stock returns, ranged from $-9,022$ to 943. The values for the IQR procedure were $Q3 = -0.65$, $Q1 = -1.58$, $IQR = 0.93$, $UB = 2.14$, and $LB = -4.37$. This procedure removed 2.43% of all of our cross-sectional data, but left us with very stable Z-score values for 97.57% of our stock data. Some researchers might use windzorization techniques. The idea of windsorization is to compute the Z-scores for all stocks, then force the stocks with signals of Z-score greater that absolute value of 3, we force them to be 3 or -3. Then recompute the Z-scores for remaining stocks. Although this seems satisfactory, it does not really work, because even after the cut off, there may be still a group of outliers, thus an iterative process is needed. This can take time and can be less than efficient.

§These are described in section 4.4.

3.3. *The risk models*

Crucial to the portfolio management process is the use of a risk model for the securities. In order to understand how crowding occurs in the financial system from the portfolio construction process, we used the leading risk models in the industry to build portfolios.¶

Most professional money managers use standard third-party risk models to manage their portfolios. The most well-known risk models are that of MSCI-Barra, Northfield, and Axioma.‖ In this paper, we use the Barra US Equity Model (USE4) which has been active since 30 June 1995.†† We also use Northfield's US Fundamental Equity Risk Model which has been active since 30 January 1990.‡‡ Finally, we use Axioma's Robust Risk Model for the US which has been active since 4 January 1982.§§ Barra is believed to lead most providers with around a 50% market share.

All of these risk models are multi-factor models. That is, these factor models assume that asset returns can be modelled as a linear combination of common risk factors (Ross 1976, Chincarini and Kim 2006). The three risk models differ by the factors chosen and other estimation techniques. We use the three prominent risk models in the industry to reconstruct the variance–covariance matrix of asset returns that real-world portfolio managers would be using to build their optimal portfolios so that we can get an accurate estimation of the crowding that may or may not occur through the portfolio construction process.

4. Transaction cost models

Transaction costs are broken down into two categories. These include fixed costs (or those easily observed in the market

¶In order to allow for comparisons across risk models, we match all data across risk model providers and the top 2000 stocks by market capitalization every month of the analysis. We matched the data by CUSIP identification.

‖The majority of asset managers use either Barra, Northfield, or Axioma and thus are a very representative group (Fabozzi *et al*. 2007, Fabozzi and Markowitz 2011). Other providers include APT and R-squared. APT's Market Risk Model for the US has been active since January 2000. For more info, see http://www.sungard.com/campaigns/fs/alternativeinvestments/apt/solutions/apt_market_risk_models.aspx. R-Squared Customized Hybrid Risk Model (CHRM) has been active since June 29th, 2007. For more info, see http://www.rsquaredriskmanagement.com/Customised-Hybrid-Risk-and-Return-Models.

††For more info, see http://www.msci.com/products/portfolio_management_analytics/equity_models/barra_us_equity_model_use4.html. BARRA has another popular risk model , the Barra US Equity Model (USE3), which has been active since 1973.

‡‡For more info, see http://www.northinfo.com/documents/8.pdf.

§§For more info, see http://axioma.com/robust.htm.

place) and variable costs (those that are less observable and therefore require more modelling to estimate).

Fixed transaction costs will typically include a per share commission that the manager must pay to the broker to execute the trades. It is reasonable to estimate a cost of \$0.005 per share for commissions. Additionally, there is the issue of the bid/offer spread. Since we know that market makers make money on trades through the spread, it is reasonable to assume that a manager will pay the offer on purchases and receive the bid for sales. These costs are assumed to be more or less constant costs that do not vary much with the size of the trade.

Variable transaction costs will typically include an estimate of the likely impact of the size of the trade on the price. There is a positive relationship between the size of a trade and market impact. However, the relationship is not always linear. As trades increase in size up to and beyond a certain threshold, the estimated market impact will increase at an increasing rate. For example, a trade that is two times the average daily trading volume (ADTV) is likely to have more than two times the market impact as a trade that is equal to the ADTV.

Transaction costs have the potential to contribute to crowding. Whilst the fixed costs are not likely to vary with the size of the trade, the variable costs will vary and can affect the final positions. For large portfolios, the transaction costs for assets with low ADTV are likely to be high and the positive gross alphas will become smaller (and possibly even negative) after transaction costs. The final portfolios are less likely to include these names. For assets with higher ADTV, more of the positive gross alpha will translate into positive net alpha and we are more likely to see these names included in the final portfolios. When multiple managers use the same transaction cost model, there may be crowding in liquid assets.

4.1. Model 1

In order to study the impacts of the transaction costs or market impact models on crowding, we use two market impact models. The first model is a structural model estimated from US equity data (Almgren *et al.* 2005). The model for market impact on trading is given by:†

$$c_{it} = \frac{I}{2} + \operatorname{sgn}(n_{it})\eta\sigma_{it}\left|\frac{n_{it}}{V_{it}T}\right|^{3/5} \tag{14}$$

where $I = \gamma\sigma_{it}\frac{n_{it}}{V_{it}}\left(\frac{N_{it}}{V_{it}}\right)^{1/4}$, $\gamma = 0.314$, $\eta = 0.142$, σ_{it} is the daily volatility of stock return i at the beginning of month t, N_{it} is the total amount of shares outstanding in the security, V_{it} is the average daily trading volume of the stock (shares traded, not dollars traded), sgn() is a function that is -1 if shares are being sold and 1 is shares are being bought, T is the time interval in which the trade takes place in number of days, for this paper we use $T = 1$, and n_{it} represents the number of shares of the security the portfolio is trading.‡

†For the purposes of this paper, the preciseness of the transaction cost model is not crucial. Any model of the form, $c_{it} = \frac{s}{2} + \frac{n_{it}}{V_{it}}^{\phi}\psi$ will be sufficient, where s is the bid-ask spread and ϕ and ψ are parameters that need to be estimated.

‡The parameter symbols have been changed from the original paper so as to be more consistent with symbols in this paper.

4.2. Model 2

The second model is the Northfield model for transaction costs which has been available since March 2009 (see Northfield (2015)). This model of market impact is estimated every month by Northfield with dynamically generated parameters for each stock. The model is of the form,

$$c_{it} = B_{it}|n_{it}| + C_{it}|n_{it}|^{0.5} \tag{15}$$

where B_{it} and C_{it} are parameters estimated by Northfield, n_{it} is the number of shares to be purchased for security i in month t, and c_{it} is expressed in terms of percentage price movement.§

4.3. Spreads

For both market impact models, we add the percentage spread cost of trading, by adding a term equal to the bid-ask spread divided by 2 divided by the current stock price multiplied by 100. Thus, the final transaction cost model is given by,

$$tc_{it} = C_{it} + \left|\frac{100s_{it}/2}{p_{it}}\right| + |c_{it}| \tag{16}$$

where C_{it} is the percentage commission cost from the trade, s_{it} is the bid-ask spread of stock i at time t, p_{it} is the price of stock i at time t, and c_{it} is the market-impact costs for stock i at time t based on one of the two market impact models. These transaction costs, tc_{it} are in percentage points. Since commission costs are typically small for institutional investors and typically a constant value, they will not be important for the analysis in this paper, thus we will assume they are zero throughout the paper.

For example, for December 2013, take two stocks, AT&T (Ticker Symbol: T), a very liquid stock, and AGL Resources (Ticker Symbol: GAS), a less liquid stock. AT&T for this particular period had a market capitalization of \$183 billion, a stock price of \$35.16, and a 10-day average daily trading volume of 18,930,000 shares. The spread was 1 cent or a 0.0284% spread. The trading costs in percentage terms for a 1% position in a \$500 million portfolio was 0.0232%. That is, a \$5 million trade of AT&T representing 142,000 shares would cost the trader \$1,160. This does not represent commissions, it is simply the market impact and spread costs. AGL Resources for this particular period had a market capitalization of \$5.6 billion, a stock price of \$47.23, and a 10-day average daily trading volume of 490,000 shares. The spread was 2 cents or a 0.0423% spread. The trading costs in percentage terms for a 1% position in a \$500 million portfolio was 0.1621%. That is, a \$5 million trade of AGL Resources representing 105,865 shares would cost the trader \$8,105.

4.4. Approximation of transaction costs for optimization model

The transaction cost models used in this paper are difficult to use in a standard optimization framework. In the case of transaction cost model 1, it is not usable even in the leading software provider platforms for portfolio optimization, like

§Northfield prefers to use the symbol S_{it} to represent the shares traded.

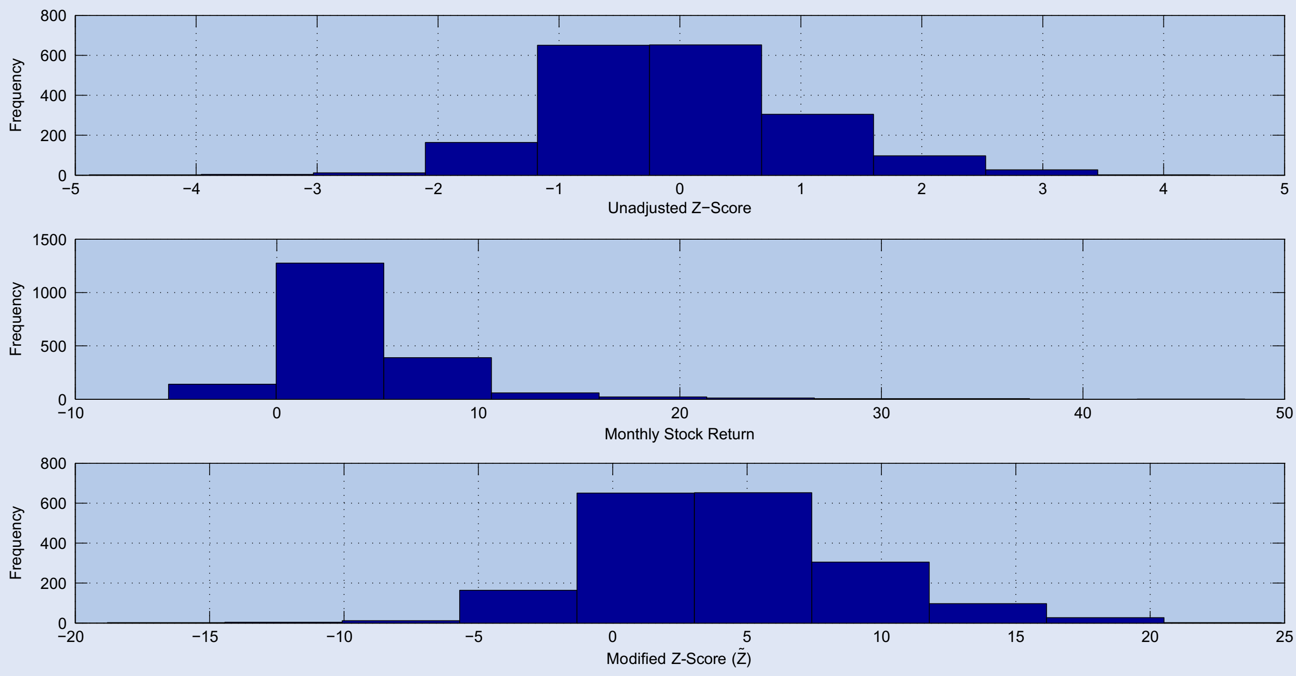

Figure 1. Conversion of Z-scores to Modified Z-scores using December 2013 data. In this figure, $\mu_r = 4.22$, $\sigma_r = 4.71$, $\mu_Z = 0$, $\sigma_Z = 1$, and $\mu_{\tilde{Z}} = 4.22$, and $\sigma_{\tilde{Z}} = 4.71$. The means and standard deviations are from a cross-section of monthly stock returns.

Axioma, Northfield, and BARRA.¶ In this paper, we present an easy-to-execute and remarkably reliable approximation to transaction costs, which could prove useful for practitioners needing to deal with a variety of transaction cost models. First, the transaction costs are computed for each stock in the portfolio by varying the portfolio weight of each stock from zero to 0.10 (the maximum possible value for any stock in the portfolio) for each net asset level. Second, a regression is run on each stock of the following form:

$$\widetilde{tc}_{it} = a_{it} w_{it} + b_{it} w_{it}^2 \tag{17}$$

where $\widetilde{tc}_{it}$ is a vector of net transaction costs from the transaction model corresponding to each stock's particular weight, a_{it} and b_{it} are parameters estimated from the linear regression.† That is, $\widetilde{tc}_{it}$ represents the percentage transaction cost of each stock multiplied by the stock's weight, w_{it}, representing the net transaction cost impact of each stock at each weight to the entire portfolio.

This approximation model works extremely well for all stocks.‡ For example, the maximum and minimum $\bar{R}^2$ for all stocks in December 2013 is 1 and 0.9998, respectively. The approximate cost model works very well at estimating the transaction costs of each stock. Figure 2 shows the actual transaction costs and approximate transaction costs for AT&T for December 2013. The approximation is excellent with $\hat{\alpha} = 0.0206$, $\hat{\beta} = 0.5096$, and $\bar{R}^2 = 0.9999$. Figure 3 shows the actual and approximate transaction costs for AGL Resources. The approximation is excellent with $\hat{\alpha} = 0.0697$, $\hat{\beta} = 0.1110$, and $\bar{R}^2 = 0.9999$.

5. Empirical simulation

5.1. Methodology

Given these realistic portfolio construction techniques described in the previous sections, we constructed 100 optimal portfolios for every risk model and for every month in our sample period from a security universe of the largest 2000 publicly traded stocks in the United States over the period 2006–2013.§ In every month of the sample, each portfolio was constructed from 2000 random alpha signals. That is, for every one of the 100 portfolios constructed, 2000 random alpha signals were created. In addition to the random alpha signals, we also created a universe of factor portfolios in which one-third of the portfolio alphas were generated based on the value factor, one-third were generated based on the momentum

¶When we began working on this paper, we considered partnering with the research staff at Axioma and the research staff of other commercial portfolio optimization software providers to alleviate the work load. However, they informed us that their systems would have trouble incorporating certain transaction cost models, like transaction cost model 1. BARRA and Bloomberg also do not support such a functional form, although they can be tweeked to approximate the costs within a given range of trade size.

†There is a different a_{it} and b_{it} for every net asset level, since the transaction costs of each stock vary with assets under management.

‡The Weierstrass approximation theorem states that every continuous function defined on a closed interval [a, b] can be uniformly approximated as closely as desired by a polynomial function. In the case of the transaction costs functions used in this paper, a quadratic function is sufficient for a very good approximation. This is achieved by transforming the concave transaction cost functions into convex functions by multiplying transaction costs by the weights, w. That is, $\widetilde{tc}_{it} = w \cdot tc_{it}$. This is done in order to construct an approximate transaction cost function that will work with a slightly modified quadratic optimization problem with quadratic constraints.

§This security universe was updated every month in our sample period between 2006 and 2013.

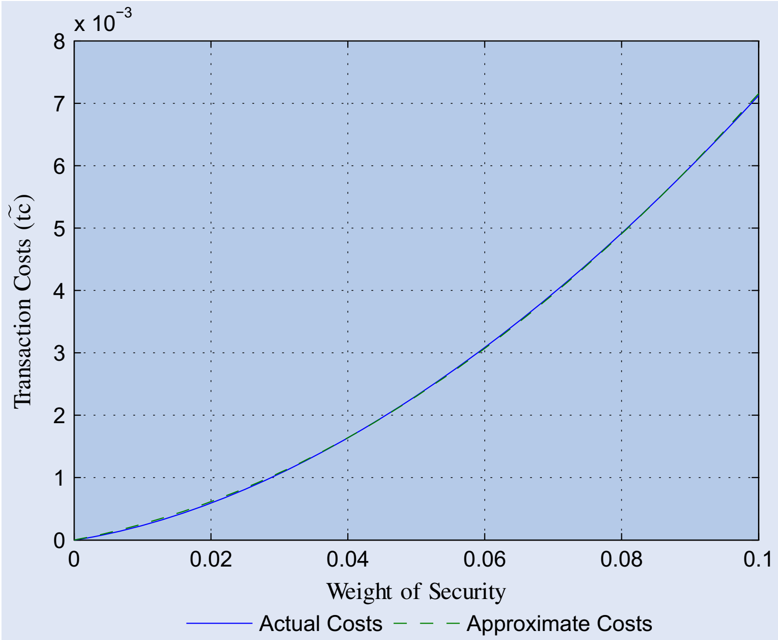

Figure 2. Actual trading costs with approximate trading costs for AT&T. This figure shows the trading costs, $\widetilde{tc}$, using transaction cost model 1 for December 2013.

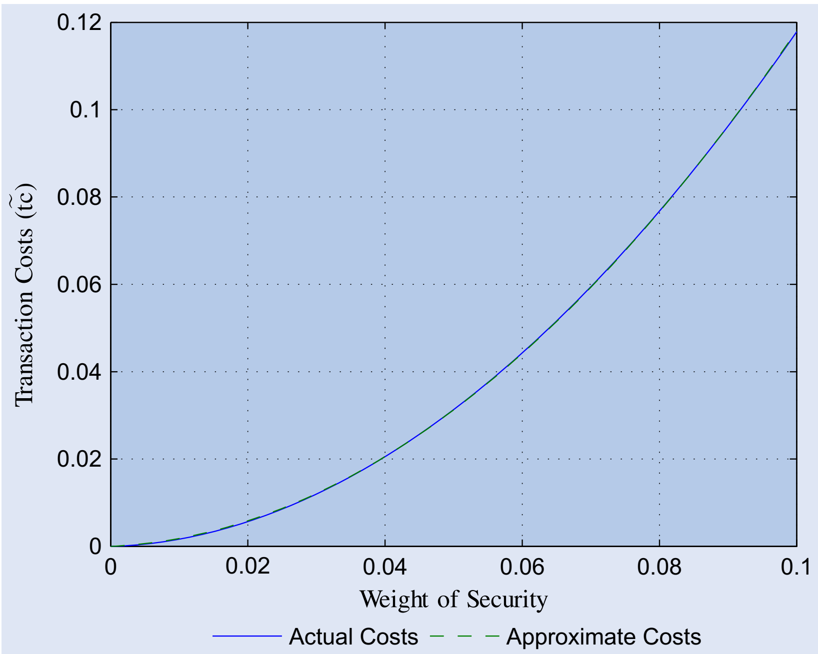

Figure 3. Actual trading costs with approximate trading costs for AGL Resources. This figure shows the trading costs, $\widetilde{tc}$, using transaction cost model 1 for December 2013.

factor, and one-third were generated based on the low-beta factor.

For each group of portfolios, four market scenarios were considered. In scenario 1, all of the portfolios were constructed without considering transaction costs. In scenario 2, each portfolio was assumed to have a total of $500 million in assets under management (AUM). In scenario 3, each portfolio was assumed to have a total of $5 billion in AUM. In scenario 4, each portfolio was assumed to have a value of $20 billion in AUM. Since the market impact (a primary driver of transaction costs) is driven by the size of the trades, adjusting the value of the portfolios from $500 million to $20 billion in AUM allows for a comparison of how transaction costs affect crowding during the portfolio construction process. The transaction costs are estimated for every stock and the approximation parameters are re-estimated for every size scenario, since they naturally would change as the portfolio size changes. The portfolios are rebalanced once per month.† For each scenario, the optimal portfolio weights were stored for all 100 portfolios in every

†Portfolio managers may not trade as often or as strictly as the simulation does, however, the point of the research is to examine

month for the random alpha portfolios. The factor portfolios optimal weights were also stored every month. These weights were used to measure crowding in each of the four scenarios.

Our analysis of these simulated portfolios covered the period 2006–2013 for transaction cost model 1 and March 2009–2013 for transaction cost model 2.† The reason that we used a different time period for transaction cost model 2 is that it was created by Northfield in 2009 and did not exist prior to this date.

5.2. Random alpha results

Tables 1 and 2 report the crowding measures by optimization framework (e.g. Long-only portfolio), by risk model 1, 2 or 3, with and without transaction costs, and by average size of the portfolio.‡ We also show in the tables, the maximum weight in any given portfolio, the minimum weight in any given portfolio, and the average number of stocks in the portfolios that are constructed. We split the analysis into two periods since one transaction cost model only existed from March 2009 until 2013.

For both periods, 2006–2009 and 2009–2013, independent of the risk model used, as the average portfolio size increased from $500 million with no transaction costs to $5 billion with transaction costs, the average crowding declined. For example, for the period 2006–2009, using risk model 1, when long portfolios were constructed without considering transaction costs, the crowding in the system was 0.58 (see column 2, row 3 of table 1). For portfolios of size $500M, crowding declined to 0.49 (see column 2, row 5), for portfolios of size $5B, crowding dropped further to 0.42 (see column 2, row 7), and for portfolios of size $20B, crowding was 0.50 (see column 2, row 9). This same pattern can be seen for portfolios constructed with the other two risk models. In particular, for risk model 2, the respective crowding numbers are 0.60, 0.45, 0.38 and 0.43 (see column 8, rows 3, 5, 7 and 9). For risk model 3, the respective crowding numbers are 0.59, 0.46, 0.38 and 0.46 (see column 14, rows 3, 5, 7 and 9). The market neutral portfolios do not show a significant change in crowding, as most of the measures of crowding are equal to 0. The omega value of crowding shows that crowding only starts to increase for market neutral portfolios at a portfolio value of $20B (compare column 3, row 2 to column 3, row 8).

A similar pattern can be seen in table 2 covering the period 2009–2013 and using an additional transaction cost model to form portfolios. For example, for the period 2009–2013, using risk model 1, crowding declines from 0.37 without considering transaction costs to 0.29 when considering transaction costs for an average portfolio size of $5B (see column 2, and rows 3 and 7 of table 2). Even when the average portfolio size is $20B, crowding is at 0.38 (see column 2, row 9), which is about the same level as when portfolios are constructed without considering transaction costs. The same pattern emerges when using transaction cost model 2, as crowding declines to 0.34, 0.30 and 0.26 for average portfolio sizes $500M, $5B, and $20B (see column 2, rows 11,13, and 15). A similar pattern can also be seen in table 2 when using risk models 2 and 3 (see columns 8 and 14). Thus, for both market neutral and long-only portfolios, crowding declines as the average portfolio grows in size from $500M to $5B.

Crowding amongst managers only starts to increase when the average size of the portfolio moves from $5 billion to $20 billion. Even in this case, crowding is greater when transaction costs are not considered at all compared to when they are considered with portfolios of size $20 billion (see columns 2, 8 and 14 of tables 1 and 2). For example, from 2006 to 2009, for risk model 1 and long-only portfolios, crowding is 0.58 with $500 million dollar portfolios and no transaction costs and 0.50 with portfolios of $20 billion in size when considering transaction costs. For risk model 2 and risk model 3, the numbers are 0.60 and 0.43 and 0.59 and 0.46, respectively. The qualitative results are similar for the period 2009–2013 and also similar for both transaction cost model 1 and 2. Thus, the average crowding for long portfolios declines as portfolios get large, even though transaction costs are increasing.

In order to establish statistical significance, we also tested whether the average crowding with transaction costs was significantly different than when not considering transaction costs. In almost all cases, the average crowding when considering transaction costs was significantly lower than not considering transaction costs at the 99% confidence level. In tables 1 and 2, this is indicated by *** (99% confidence level) and ** (95% confidence level). We also tested whether the average crowding from portfolios with $20 billion and hence more transaction costs was significantly smaller than portfolios with $500 million in assets, and found that it was for risk models 2 and 3 of the long-only portfolios during the period 2006–2009, but only for risk model 2 during the period 2009–2013.

The crowding measure for market neutral portfolios was generally much smaller than for long-only portfolios. In fact, in most cases, to two decimal points, it was 0.00. Thus, in order to examine the effect of transaction costs on market neutral portfolios, we focused mainly on the measure of omega (Ω). Omega measures the crowding of the constructed portfolio to the crowding of the random alpha signals. Thus, a value greater than 1 indicates that the portfolio construction process led to more crowding than the alpha signals amongst portfolios. The higher the value of omega, the more crowding that occurs from portfolio construction.

For market neutral portfolio managers of size $500 million to $5 billion, relative crowding (Ω) decreases when considering transaction costs. As the average portfolio size increases to $20 billion, relative crowding is generally higher than portfolios of smaller size for the period 2006–2009, but once again lower for the period 2009–2013. For example, for the period 2006–2009, the average relative crowding is almost double for a portfolio of size $20 billion than for portfolios that do not consider transaction costs using risk models 1 and 3 (see columns 3 and

how portfolio construction and transaction costs affect crowding and thus a controlled setting is required.

†We have data for a longer time period, but the simulations take an enormous amount of time to compute and thus we limited our sample from 2006 to 2013. For example, the 100 random alpha portfolios can take 20 days to complete one historical year of analysis when running on 12 processors in parallel.

‡The empirical testing of the crowding induced by risk models was extremely complicated. In order to dynamically simulate the portfolios, we had to create an entire program to do professional portfolio optimization. We used MATLAB 2014a and CPLEX from IBM through the MATLAB API to perform the empirical analysis.

Table 1. Summary of crowding from random alpha models and transaction costs from 2006 to February 2009.

	Risk model 1						Risk model 2						Risk model 3					
	C	Omega	SR	Max	Min	$\bar{N}$	C	Omega	SR	Max	Min	$\bar{N}$	C	Omega	SR	Max	Min	$\bar{N}$
Alpha	−0.00																	
Long-only																		
MN NTC	−0.00	0.75	−3708.352	0.004	−0.004	645	−0.00	0.84	−2437.77	0.005	−0.005	611	0.00	0.50	−3296.92	0.006	−0.01	632
LONG NTC	0.58	−141.26	−140.911	0.076	0.000	63	0.60	−181.90	−175.48	0.072	0.000	75	0.59	−156.62	−184.22	0.079	0.00	64
Port. Size ($500M)																		
MN TC1	−0.00	0.27	−8.171	0.007	−0.006	567	0.00	−0.04	−7.84	0.006	−0.006	543	0.00	0.11	−7.49	0.009	−0.01	556
LONG TC1	0.49	−127.77	−0.512	0.079	0.000	67	0.45**	−123.77	−1.00	0.071	0.000	89	0.46**	−116.86	−0.84	0.080	0.00	71
Port. Size ($5B)																		
MN TC1	0.00	0.63	−15.027	0.007	−0.007	527	0.00	0.10	−13.88	0.010	−0.011	514	0.00	0.47	−13.98	0.009	−0.01	519
LONG TC1	0.42***	−91.04	−1.427	0.077	0.000	102	0.38***	−113.74	−1.59	0.072	0.000	138	0.38***	−111.11	−1.71	0.077	0.00	114
Port. Size ($20B)																		
MN TC1	0.00	1.42	−21.240	0.013	−0.013	157	0.00	0.09	−20.03	0.014	−0.014	456	0.00	1.13	−20.05	0.014	−0.01	460
LONG TC1	0.50	294.63	−2.152	0.072	0.000	157	0.43***	151.19	−2.26	0.064	0.000	217	0.46***	241.19	−2.33	0.072	0.00	176

Notes: This table presents various crowding measures from the constructed portfolios using various portfolio optimization structures that minimize volatility using various risk models over the period 2006 to February 2009. Risk Model 1, 2 and 3 represent leading risk models used in the industry. The names are purposely omitted so as to not identify any particular risk model. All numbers in the figure are averages of various variables constructed from monthly portfolios. The computations are based on 100 portfolios formed from random alpha signals. C represents our crowding measure as described in the paper, $C = \frac{\sum_{i=1}^{M}\sum_{j=1}^{M} S_{p:i,j} - M}{M^2-M}$. Ω measures the relative crowding between random signals and actual portfolios, $\Omega = \frac{\sum_{i=1}^{M}\sum_{j=1}^{M} S_{p:i,j} - M}{\sum_{i=1}^{M}\sum_{j=1}^{M} S_{\alpha:i,j} - M}$. A higher value means that risk model creates more crowding. $S.R.$ is a pseudo-Sharpe ratio for each portfolio defined as the portfolio's annualized forward one-month return minus transaction costs divided by its ex-ante standard deviation. Max represents the median of the maximum weight of any portfolio over all months, Min represents the median of the minimum weight of any portfolio over all months, and N represents the average number of stocks across portfolios in any given month over all months. The optimizations represent optimizations which attempt to minimize the variance of the portfolio subject to a target alpha subject to various constraints as explained in the paper. Two transaction cost models are considered, model 1 (TC1) and model 2 (TC2). The transaction costs include spreads and market impact of the following form: $tc_{it} = \left|\frac{100 s_{it}/2}{p_{it}}\right| + |c_{it}|$, where s_{it} is the bid-ask spread of stock i at time t. For model 1, $c_{it} = \frac{I}{2} + \text{sgn}(n_{it})\eta\sigma_{it}\left|\frac{n_{it}}{V_{it}T}\right|^{3/5}$, where $I = \gamma\sigma_{it}\frac{n_{it}}{V_{it}}\left(\frac{N_{it}}{V_{it}}\right)^{1/4}$, $\gamma = 0.314$, $\eta = 0.142$, σ_{it} is the daily volatility of stock return i at the beginning of month t, N_{it} is the total amount of shares outstanding in the security, V_{it} is the average daily trading volume of the stock, T is the time interval in which the trade takes place in number of days, for this paper we use $T = 1$, and n_{it} represents the number of shares of the security the portfolio is trading. For model 2, $c_{it} = B_{it}|n_{it}| + C_{it}|n_{it}|^{0.5}$, where B_{it} and C_{it} are parameters estimated by Northfield, n_{it} is the number of shares to be purchased for security i in month t, and c_{it} is expressed in terms of percentage price movement. Model 2 exists only since March 2009. ***, ** indicates the a 99 and 95% significant difference respectively in the average crowding from this portfolio and a portfolio that doesn't consider transaction costs. MN is for the market neutral portfolios and LONG is for the long portfolios.

Table 2. Summary of crowding from random alpha models and transaction costs from March 2009 to 2013.

	Risk model 1						Risk model 2						Risk model 3					
	C	Omega	SR	Max	Min	$\bar{N}$	C	Omega	SR	Max	Min	$\bar{N}$	C	Omega	SR	Max	Min	$\bar{N}$
Alpha	0.00																	
Long-only																		
MN NTC	−0.00	27.11	−42.646	0.006	−0.006	747	−0.00	55.14	−60.02	0.008	−0.008	731	−0.00	62.98	−32.70	0.008	−0.01	736
LONG NTC	0.37	−19958.74	−1.123	0.090	0.000	44	0.41	−24910.52	−8.03	0.087	0.000	48	0.39	−21522.80	−4.13	0.092	0.00	46
Port. Size ($500M)																		
MN TC1	−0.00	26.23	−7.132	0.007	−0.008	663	−0.00	27.70	−6.66	0.009	−0.009	652	−0.00	57.83	−6.51	0.010	−0.01	652
LONG TC1	0.32	−17843.58	0.794	0.090	0.000	50	0.27***	−16434.66	0.32	0.085	0.000	66	0.31**	−16026.43	0.53	0.091	0.00	56
Port. Size ($5B)																		
MN TC1	−0.00	26.74	−14.669	0.009	−0.009	594	−0.00	19.87	−13.70	0.012	−0.012	588	−0.00	51.50	−13.61	0.012	−0.01	586
LONG TC1	0.29***	−15581.70	0.167	0.088	0.000	77	0.25***	−13758.30	−0.08	0.082	0.000	106	0.27***	−14462.44	−0.04	0.088	0.00	92
Port. Size ($20B)																		
MN TC1	−0.00	40.62	−22.275	0.013	−0.013	122	−0.00	27.97	−20.73	0.015	−0.016	503	−0.00	45.28	−21.27	0.015	−0.02	502
LONG TC1	0.38	−6862.70	−0.491	0.082	0.000	122	0.32***	−7217.42	−0.66	0.075	0.000	159	0.36	−6539.03	−0.66	0.082	0.00	144
Port. Size ($500M)																		
MN TC2	−0.00	29.83	−7.346	0.007	−0.007	655	−0.00	45.16	−6.44	0.009	−0.009	644	−0.00	60.42	−6.73	0.010	−0.01	644
LONG TC2	0.34	−17964.16	0.960	0.091	0.000	43	0.33***	−20577.48	0.51	0.088	0.000	47	0.33	−16635.14	0.80	0.093	0.00	44
Port. Size ($5B)																		
MN TC2	−0.00	28.76	−13.438	0.009	−0.009	591	−0.00	30.47	−11.91	0.011	−0.011	583	−0.00	58.93	−12.31	0.011	−0.01	581
LONG TC2	0.30**	−16088.35	0.558	0.091	0.000	45	0.27***	−16026.43	0.06	0.086	0.000	52	0.28***	−14002.91	0.30	0.092	0.00	47
Port. Size ($20B)																		
MN TC2	−0.00	28.39	−19.124	0.010	−0.010	513	−0.00	1.88	−17.08	0.013	−0.013	506	−0.00	56.05	−17.56	0.014	−0.01	506
LONG TC2	0.26***	−14485.43	0.157	0.091	0.000	49	0.22***	−12348.61	−0.25	0.086	0.000	59	0.24***	−13175.77	−0.12	0.091	0.00	53

Notes: This table presents various crowding measures from the constructed portfolios using various portfolio optimization structures that minimize volatility using various risk models over the period March 2009–2013. Risk Model 1, 2 and 3 represent leading risk models used in the industry. The names are purposely omitted so as to not identify any particular risk model. All numbers in the figure are averages of various variables constructed from monthly portfolios. The computations are based on 100 portfolios formed from random alpha signals. C represents our crowding measure as described in the paper, $C = \frac{\sum_{i=1}^{M}\sum_{j=1}^{M} S_{p:i,j} - M}{M^2 - M}$. Ω measures the relative crowding between random signals and actual portfolios, $\Omega = \frac{\sum_{i=1}^{m}\sum_{j=1}^{m} S_{p:i,j} - m}{\sum_{i=1}^{m}\sum_{j=1}^{M} S_{\alpha:i,j} - M}$. A higher value means that risk model creates more crowding. $S.R.$ is a pseudo-Sharpe ratio for each portfolio defined as the portfolio's annualized forward one-month return minus transaction costs divided by its ex-ante standard deviation. Max represents the median of the maximum weight of any portfolio over all months, Min represents the median of the minimum weight of any portfolio over all months, and N represents the average number of stocks across portfolios in any given month over all months. The optimizations represent optimizations which attempt to minimize the variance of the portfolio subject to a target alpha subject to various constraints as explained in the paper. Two transaction cost models are considered, model 1 (TC1) and model 2 (TC2). The transaction costs include spreads and market impact of the following form: $tc_{it} = \left|\frac{100 s_{it}/2}{p_{it}}\right| + |c_{it}|$, where s_{it} is the bid-ask spread of stock i at time t. For model 1, $c_{it} = \frac{I}{2} + \text{sgn}(n_{it})\eta\sigma_{it}\left|\frac{n_{it}}{V_{it}T}\right|^{3/5}$, where $I = \gamma\sigma_{it}\frac{n_{it}}{V_{it}}\left(\frac{N_{it}}{V_{it}}\right)^{1/4}$, $\gamma = 0.314$, $\eta = 0.142$, σ_{it} is the daily volatility of stock return i at the beginning of month t, N_{it} is the total amount of shares outstanding in the security, V_{it} is the average daily trading volume of the stock, T is the time interval in which the trade takes place in number of days, for this paper we use $T = 1$, and n_{it} represents the number of shares of the security the portfolio is trading. For model 2, $c_{it} = B_{it}|n_{it}| + C_{it}|n_{it}|^{0.5}$, where B_{it} and C_{it} are parameters estimated by Northfield, n_{it} is the number of shares to be purchased for security i in month t, and c_{it} is expressed in terms of percentage price movement. Model 2 exists only since March 2009. ***, ** indicates the 99% and 95% significant difference respectively in the average crowding from this portfolio and a portfolio that doesn't consider transaction costs. MN is for the market neutral portfolios and LONG is for the long portfolios.

15 of table 1). However, for risk model 2, crowding declines. For the 2009–2013 period and transaction cost model 1, relative crowding generally declines as the portfolio grows from $500 million without considering transaction costs to $20 billion. For risk models 2 and 3, the no transaction cost measure of relative crowding is 55 and 63 compared to 28 and 45 using transaction cost model 1 (see columns 9 and 15 and rows 2 and 8). Only for risk model 1 is relative crowding greater for the $20 billion dollar average portfolio compared to the $500 million dollar average portfolio without transaction costs (see column 3, rows 3 and 8). Similar implications come from using transaction cost model 2 over the period 2009–2013.

We also tested for the statistical significance of the differences in average crowding for market neutral portfolios. We found no statistical difference in the average crowding. Thus, although average crowding generally declines with transaction costs for market neutral portfolios, the lack of significance implies that transaction costs probably did not play a large role in crowding at these asset levels.

Figures 4–6 show the way crowding changes over time. For long-only portfolios, for most of the period between 2006 and 2013, the crowding from the largest portfolios considering transaction costs (red line) is lower than that of the smaller portfolios that do not even consider transaction costs (blue dotted line). However, in 2006 and in 2013, this was reversed (see figure 4). This general pattern seems to be true for all risk models used (see figure 6). One will also notice that crowding amongst the portfolios started to increase just prior to the quant crisis of August 2007, and continued to increase after the crisis reaching a peak at about the time of the Lehman Bankruptcy in September of 2008. There is also a spike in crowding for portfolios constructed with an average size of $20B between October and December 2010. It is unclear what may have caused this, but it may have something to do with diminished liquidity in many available equity securities.

Figure 5 shows that for most of the time, the relative crowding for market neutral models without transaction costs is inline with large portfolios that consider transaction costs, except for certain brief periods, where the relative crowding of large portfolios increases enormously. These particular periods are driving the average results discussed in the tables.

5.3. *Factor model results*

Tables 3 and 4 report the crowding measures by optimization framework (e.g. Long-Only Portfolio), by risk model 1, 2 or 3, with and without transaction costs, for the three realistic alpha portfolios. One-third of the portfolios used the value factor for alpha signals, one-third used the momentum factor for alpha signals, and one-third used the low-beta factor for alpha signals.† The results are qualitatively similar to what we found earlier.

For long-only portfolios, crowding is less for portfolios that are optimized to explicitly consider transaction costs than those that do not. This is true for both the 2006–2009 period and the 2009–2013 period, with small exceptions. For example, for the 2006–2009 period, the crowding amongst portfolios due to the alpha models is 0.09 (see column 2, row 1 of table 3). For long-only portfolios, this crowding increased to 0.52 without considering transaction costs. It declined to 0.48 and 0.46 when the average portfolio size was $500M or $5B (see column 2, rows 5 and 7). For other risk models, the crowding declined from 0.42 to 0.37 for risk model 2 and 0.55 to 0.44 for risk model 3 for portfolios of average size $5B (see columns 8 and 14 and rows 3 and 7).

A similar pattern of declining crowding with transaction costs is found when studying the period 2009–2013 (see table 4). For example, for risk model 1, the crowding for long-only portfolios without transaction costs was 0.40, but 0.41 and 0.44 for portfolios of size $500M and $5B (see column 2, rows 3, 5, and 7 of table 4). With average portfolio size of $20B, crowding did start to increase to 0.52. For risk model 3 and transaction cost model 2, crowding declined even when the average portfolio size was $20B (see column 1, rows 11, 13, and 15). This pattern is similar for risk models 2 and 3 for long-only portfolios. In these cases, the decline in crowding was statistically significant in many cases. For example, for risk model 3 and transaction cost model 2, crowding declined from 0.46 to 0.38 with an average portfolio size of $5B (see column 14, rows 3 and 13) and was statistically significant at the 95% confidence level.

For the market neutral portfolios, the point estimates of crowding slightly deviated from zero. For the period 2006–2009, most of the crowding changes from using transaction cost models were insignificant. Thus, transaction cost models did not seem to increase or decrease crowding. One exception was for $20B portfolio sizes for risk model 3 and transaction cost model 2. In this particular case, crowding actually changed quite a bit. Over the 2009–2013 period, using risk model 3, the crowding without transaction costs is 0.02, however, with a portfolio of size $500M and $5B using transaction cost model 2, it significantly increased to 0.03 (see column 14, rows 2, 10, and 12). This can also be seen using omega which changes for the same scenario from 0.28 to 0.49 and 0.50, respectively. Whilst crowding seems to increase with transaction costs for some market neutral situations, it doesn't for others. Also, the magnitude of this increase is rather small; a crowding increase in from 0.02 to 0.03 is large in percentage terms but not in absolute terms when considering that crowding ranges from −1 to 1.

Overall, the evidence when considering portfolios constructed from realistic factor models, including a value factor, a momentum factor and a beta factor, is that transaction cost models do not seem to increase crowding. Rather, if transaction cost models were properly integrated into the portfolio construction process, they would decrease crowding up to portfolios of average size $20 billion.

5.4. *Implications*

The crowding of investment in securities can lead to similar positions by similar investors that may eventually lead to a cascade when investors must rebalance their portfolios. Rebalancing cascades might occur when investors follow similar

†These results add to the previous analysis by examining how alpha crowding plays into the transaction cost impact. One suggestion for further work is to create a universe of portfolios managed from 100 or 1000 correlated factor models, but individually distinct.

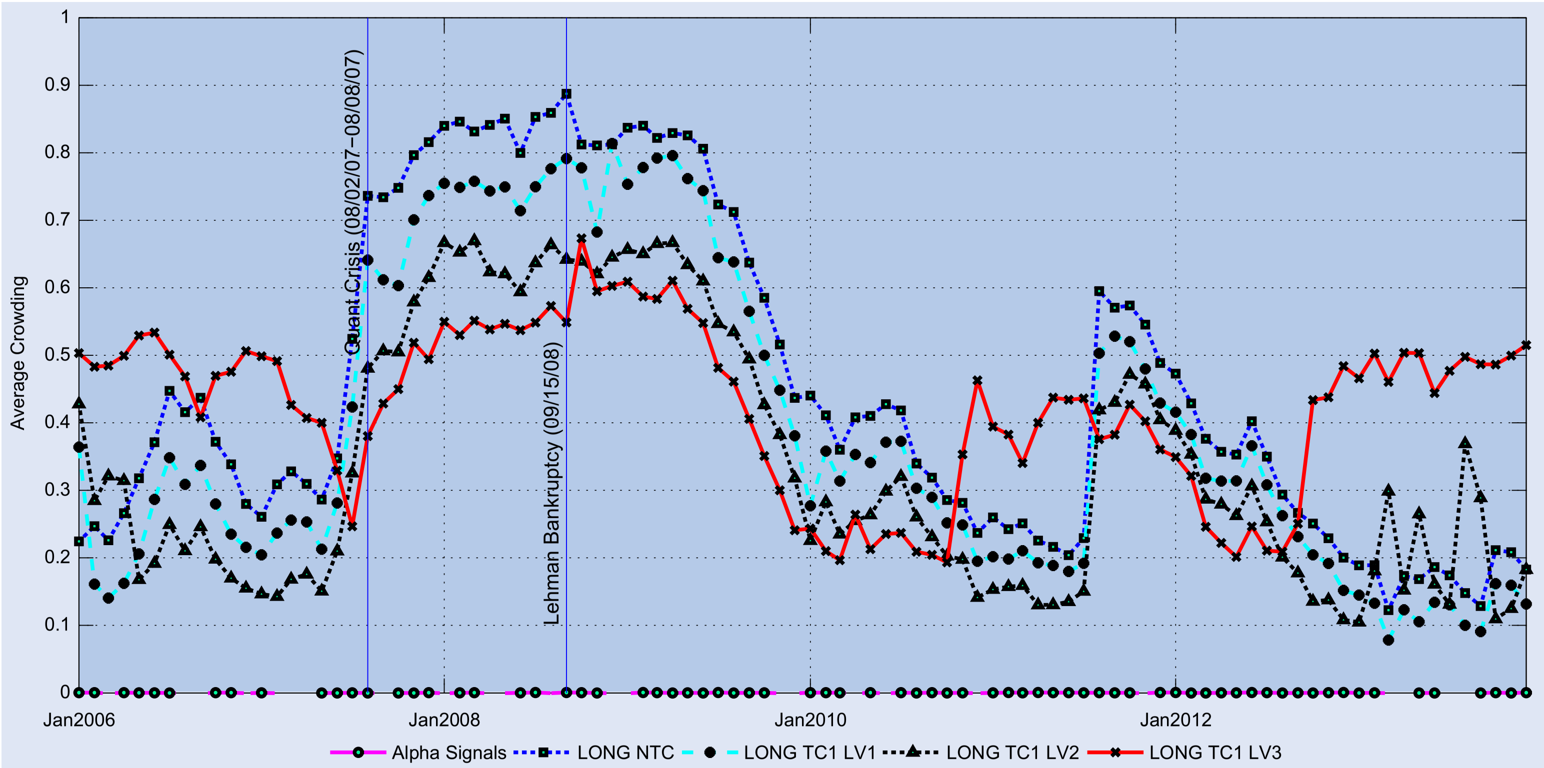

Figure 4. Crowding over time for random alpha models' long portfolios. This figure represents the crowding measures for long-only portfolios constructed every month from 2006 to 2013 using three industry risk models. At every point in time, actual data was used to construct 100 random alpha signal vectors and consequently 100 optimized portfolios. Two transaction cost models are considered, model 1 (TC1) and model 2 (TC2). These graphs only show the crowding numbers for portfolios created with model 1. The transaction costs include spreads and market impact of the following form: $tc_{it} = \left|\frac{100 s_{it}/2}{p_{it}}\right| + |c_{it}|$, where s_{it} is the bid-ask spread of stock i at time t. For model 1, $c_{it} = \frac{I}{2} + \mathrm{sgn}(n_{it})\eta\sigma_{it}\left|\frac{n_{it}}{V_{it}T}\right|^{3/5}$, where $I = \gamma\sigma_{it}\frac{n_{it}}{V_{it}}\left(\frac{N_{it}}{V_{it}}\right)^{1/4}$, $\gamma = 0.314$, $\eta = 0.142$, σ_{it} is the daily volatility of stock return i at the beginning of month t, N_{it} is the total amount of shares outstanding in the security, V_{it} is the average daily trading volume of the stock, T is the time interval in which the trade takes place in number of days, for this paper we use $T = 1$, and n_{it} represents the number of shares of the security the portfolio is trading. LONG NTC is the monthly crowding for long-only portfolios that do not consider transaction costs, LONG TC1 LV1, LV2 and LV3 are long portfolios that consider transaction costs with portfolios of size \$500M, \$5B, and \$20B respectively.

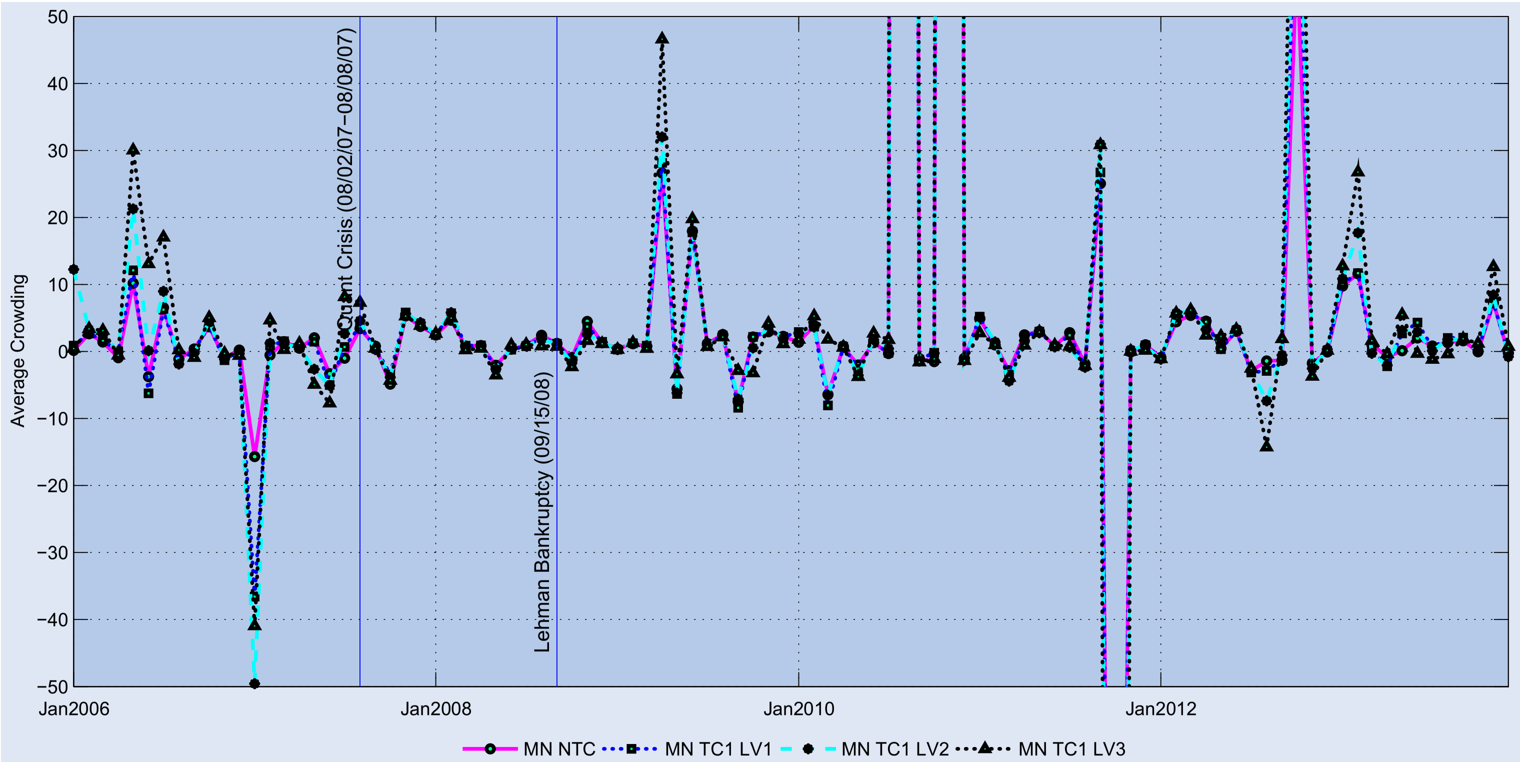

Figure 5. Crowding over time for random alpha models' market neutral portfolios. This figure represents the crowding measures for long-only portfolios constructed every month from 2006 to 2013 using three industry risk models. At every point in time, actual data were used to construct 100 random alpha signal vectors and consequently 100 optimized portfolios. Two transaction cost models are considered, model 1 (TC1) and model 2 (TC2). These graphs only show the crowding numbers for portfolios created with model 1. The transaction costs include spreads and market impact of the following form: $tc_{it} = \left|\frac{100 s_{it}/2}{p_{it}}\right| + |c_{it}|$, where s_{it} is the bid-ask spread of stock i at time t. For model 1, $c_{it} = \frac{I}{2} + \text{sgn}(n_{it})\eta\sigma_{it}\left|\frac{n_{it}}{V_{it}T}\right|^{3/5}$, where $I = \gamma\sigma_{it}\frac{n_{it}}{V_{it}}\left(\frac{N_{it}}{V_{it}}\right)^{1/4}$, $\gamma = 0.314$, $\eta = 0.142$, σ_{it} is the daily volatility of stock return i at the beginning of month t, N_{it} is the total amount of shares outstanding in the security, V_{it} is the average daily trading volume of the stock, T is the time interval in which the trade takes place in number of days, for this paper we use $T = 1$, and n_{it} represents the number of shares of the security the portfolio is trading. MN NTC is the monthly crowding for long-only portfolios that do not consider transaction costs, MN TC1 LV1, LV2 and LV3 are long portfolios that consider transaction costs with portfolios of size \$500M, \$5B and \$20B, respectively.

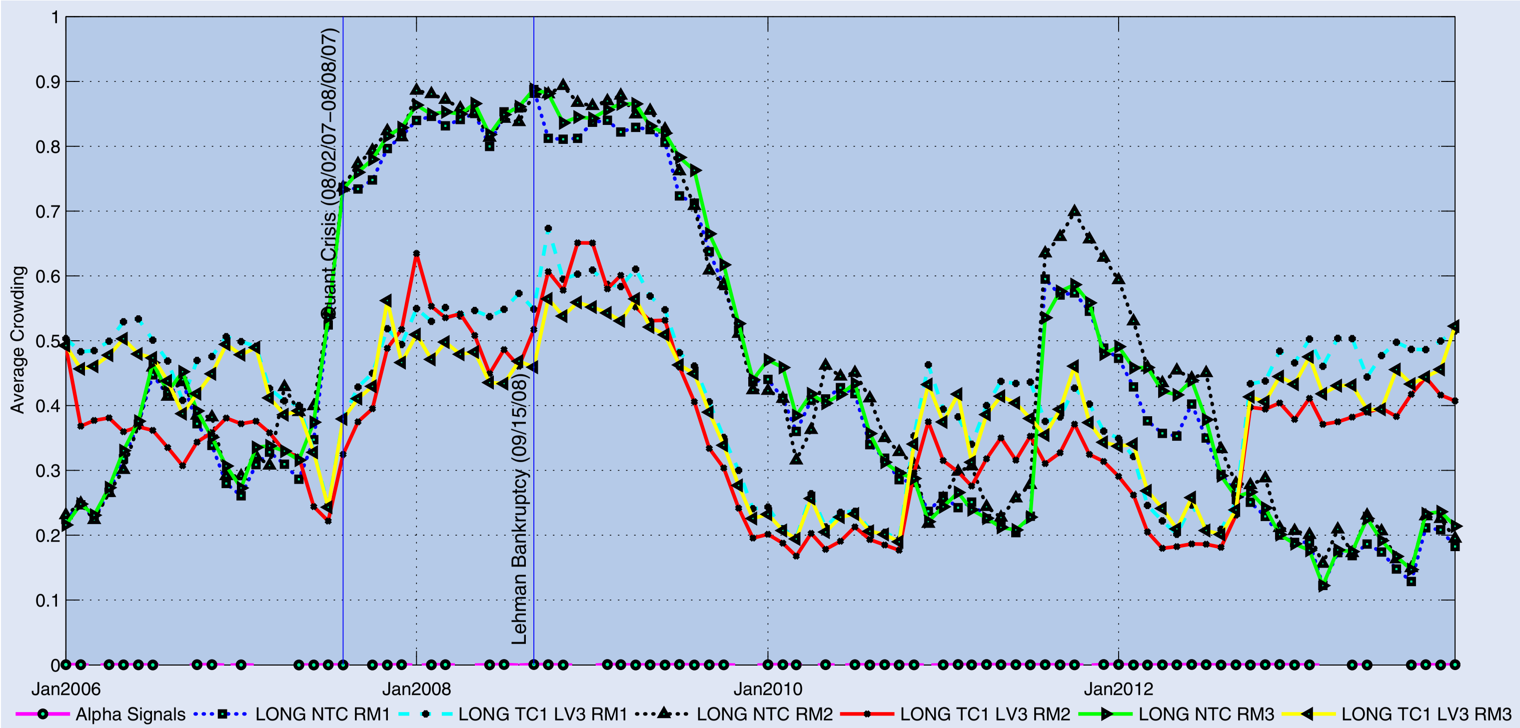

Figure 6. Crowding over time for random alpha models' long portfolios using all risk models. This figure represents the crowding measures for long-only portfolios constructed every month from 2006 to 2013 using three industry risk models. At every point in time, actual data were used to construct 100 random alpha signal vectors and consequently 100 optimized portfolios. Two transaction cost models are considered, model 1 (TC1) and model 2 (TC2). These graphs only show the crowding numbers for portfolios created with model 1. The transaction costs include spreads and market impact of the following form: $tc_{it} = \left|\frac{100 s_{it}/2}{p_{it}}\right| + |c_{it}|$, where s_{it} is the bid-ask spread of stock i at time t. For model 1, $c_{it} = \frac{I}{2} + \text{sgn}(n_{it})\eta\sigma_{it}\left|\frac{n_{it}}{V_{it}T}\right|^{3/5}$, where $I = \gamma\sigma_{it}\frac{n_{it}}{V_{it}}\left(\frac{N_{it}}{V_{it}}\right)^{1/4}$, $\gamma = 0.314$, $\eta = 0.142$, σ_{it} is the daily volatility of stock return i at the beginning of month t, N_{it} is the total amount of shares outstanding in the security, V_{it} is the average daily trading volume of the stock, T is the time interval in which the trade takes place in number of days, for this paper we use $T = 1$, and n_{it} represents the number of shares of the security the portfolio is trading. LONG NTC RM1, RM2 and RM3 represent the monthly crowding for long-only portfolios that do not consider transaction costs using risk model 1, risk model 2 and risk model 3, respectively. LONG TC1 LV3 RM1, RM2 and RM3 are long portfolios that consider transaction costs with portfolios of size \$20B using risk model 1, risk model 2 and risk model 3 respectively.

Table 3. Summary of crowding from factor models and transaction costs from 2006 to February 2009.

	Risk model 1						Risk model 2						Risk model 3					
	C	Omega	SR	Max	Min	$\bar{N}$	C	Omega	SR	Max	Min	$\bar{N}$	C	Omega	SR	Max	Min	$\bar{N}$
Alpha	0.09																	
Long-only																		
MN NTC	−0.02	−0.21	142.583	0.009	−0.005	1025	−0.00	0.05	4.13	0.014	−0.011	943	−0.01	−0.07	21.49	0.008	−0.01	1004
LONG NTC	0.52	3.48	−3.013	0.070	0.000	81	0.42	1.66	−1.87	0.069	0.000	84	0.55	3.87	−3.36	0.077	0.00	92
Port. Size ($500M)																		
MN TC1	−0.02	−0.23	−8.486	0.018	−0.010	583	−0.00	0.21	−19.79	0.020	−0.017	508	−0.01	−0.12	−9.04	0.016	−0.01	566
LONG TC1	0.48	3.29	−0.003	0.076	0.000	109	0.39	2.01	−0.00	0.064	0.000	128	0.49	3.46	−0.01	0.080	0.00	118
Port. Size ($5B)																		
MN TC1	−0.02	−0.22	−18.813	0.022	−0.014	413	−0.01	0.06	−3.58	0.025	−0.023	375	−0.03	−0.27	−10.44	0.020	−0.01	418
LONG TC1	0.46***	3.89	−0.008	0.063	0.000	145	0.37***	2.23	−0.01	0.055	0.000	191	0.44***	3.81	−0.01	0.066	0.00	153
Port. Size ($20B)																		
MN TC1	−0.03	−0.46	−30.476	0.025	−0.021	313	−0.02	−0.27	−8.43	0.026	−0.024	295	−0.06***	−0.62	6.71	0.021	−0.02	323
LONG TC1	0.51***	1.57	−0.014	0.049	0.000	179	0.43	0.69	−0.01	0.044	0.000	249	0.49***	2.18	−0.01	0.052	0.00	195

Notes: This table presents various crowding measures from the constructed portfolios using various portfolio optimization structures that minimize volatility using various risk models over the period 2006 to February 2009. Risk Model 1, 2,and 3 represent leading risk models used in the industry. The names are purposely omitted so as to not identify any particular risk model. All numbers in the figure are averages of various variables constructed from monthly portfolios. The computations are based the portfolio managers following one of three factor models, value, momentum or low beta. C represents our crowding measure as described in the paper, $C = \frac{\sum_{i=1}^{M}\sum_{j=1}^{M} S_{p:i,j} - M}{M^2 - M}$. Ω measures the relative crowding between random signals and actual portfolios, $\Omega = \frac{\sum_{i=1}^{M}\sum_{j=1}^{M} S_{p:i,j} - M}{\sum_{i=1}^{M}\sum_{j=1}^{M} S_{\alpha:i,j} - M}$. A higher value means that risk model creates more crowding. $S.R.$ is a pseudo-Sharpe ratio for each portfolio defined as the portfolio's annualized forward one-month return minus transaction costs divided by its ex-ante standard deviation. Max represents the median of the maximum weight of any portfolio over all months, Min represents the median of the minimum weight of any portfolio over all months, and N represents the average number of stocks across portfolios in any given month over all months. The optimizations represent optimizations which attempt to minimize the variance of the portfolio subject to a target alpha subject to various constraints as explained in the paper. Two transaction cost models are considered, model 1 (TC1) and model 2 (TC2). The transaction costs include spreads and market impact of the following form: $tc_{it} = \left|\frac{100 s_{it}/2}{p_{it}}\right| + |c_{it}|$, where s_{it} is the bid-ask spread of stock i at time t. For model 1, $c_{it} = \frac{I}{2} + \mathrm{sgn}(n_{it})\eta\sigma_{it}\left|\frac{n_{it}}{V_{it}T}\right|^{3/5}$, where $I = \gamma\sigma_{it}\frac{n_{it}}{V_{it}}\left(\frac{N_{it}}{V_{it}}\right)^{1/4}$, $\gamma = 0.314$, $\eta = 0.142$, σ_{it} is the daily volatility of stock return i at the beginning of month t, N_{it} is the total amount of shares outstanding in the security, V_{it} is the average daily trading volume of the stock, T is the time interval in which the trade takes place in number of days, for this paper we use $T = 1$, and n_{it} represents the number of shares of the security the portfolio is trading. For model 2, $c_{it} = B_{it}|n_{it}| + C_{it}|n_{it}|^{0.5}$, where B_{it} and C_{it} are parameters estimated by Northfield, n_{it} is the number of shares to be purchased for security i in month t, and c_{it} is expressed in terms of percentage price movement. Model 2 exists only since March 2009. ***, ** indicates the 99% and 95% significant difference respectively in the average crowding from this portfolio and a portfolio that doesn't consider transaction costs. MN is for the market neutral portfolios and LONG is for the long portfolios.

Table 4. Summary of crowding from factor models and transaction costs from March 2009 to 2013.

	Risk model 1						Risk model 2						Risk model 3					
	C	Omega	SR	Max	Min	$\bar{N}$	C	Omega	SR	Max	Min	$\bar{N}$	C	Omega	SR	Max	Min	$\bar{N}$
Alpha	0.12																	
Long-only																		
MN NTC	0.00	0.18	−6.608	0.010	−0.009	901	0.01	0.49	1.05	0.022	−0.017	809	0.02	0.28	1.16	0.009	−0.01	892
LONG NTC	0.40	4.53	−0.049	0.084	0.000	58	0.38	4.37	−0.10	0.083	0.000	54	0.46	5.80	−0.09	0.080	0.00	69
Port. Size ($500M)																		
MN TC1	0.01	0.17	−0.112	0.016	−0.012	542	0.00	0.23	−0.03	0.028	−0.022	488	0.03***	0.57	−0.04	0.014	−0.01	538
LONG TC1	0.41	3.72	0.003	0.084	0.000	76	0.34	2.62	−0.00	0.076	0.000	89	0.45	4.26	−0.00	0.085	0.00	89
Port. Size ($5B)																		
MN TC1	0.02	−0.01	−0.335	0.026	−0.016	389	−0.01***	−0.24	−0.11	0.029	−0.025	363	0.03	0.08	−0.14	0.021	−0.02	392
LONG TC1	0.44	5.43	−0.007	0.082	0.000	104	0.35	4.00	−0.01	0.071	0.000	141	0.44	5.93	−0.01	0.082	0.00	124
Port. Size ($20B)																		
MN TC1	0.00	−0.27	−0.526	0.034	−0.025	301	−0.03***	−0.80	−0.18	0.034	−0.031	290	−0.00	−0.44	−0.23	0.028	−0.02	303
LONG TC1	0.52***	7.49	−0.014	0.073	0.000	136	0.42	5.13	−0.01	0.062	0.000	190	0.52	8.25	−0.02	0.074	0.00	162
Port. Size ($500M)																		
MN TC2	0.01	0.33	−0.017	0.015	−0.012	508	0.01	0.41	−0.00	0.032	−0.024	454	0.03**	0.49	0.01	0.014	−0.01	502
LONG TC2	0.38	3.54	0.005	0.085	0.000	57	0.36	2.86	0.00	0.081	0.000	56	0.42	3.51	0.00	0.086	0.00	64
Port. Size ($5B)																		
MN TC2	0.02**	0.29	−0.080	0.019	−0.015	358	0.00	0.22	−0.03	0.035	−0.028	330	0.03***	0.50	−0.01	0.018	−0.02	359
LONG TC2	0.38	3.22	−0.000	0.084	0.000	60	0.31**	2.22	−0.00	0.078	0.000	65	0.38**	3.28	−0.00	0.085	0.00	66
Port. Size ($20B)																		
MN TC2	0.02***	0.39	−0.058	0.024	−0.018	264	−0.01***	−0.20	−0.05	0.035	−0.030	248	0.03**	0.45	0.10	0.023	−0.02	263
LONG TC2	0.35	3.85	−0.003	0.081	0.000	66	0.30***	2.88	−0.00	0.075	0.000	74	0.36***	3.93	−0.00	0.083	0.00	71

Notes: This table presents various crowding measures from the constructed portfolios using various portfolio optimization structures that minimize volatility using various risk models over the period March 2009 to 2013. Risk Model 1, 2 and 3 represent leading risk models used in the industry. The names are purposely omitted so as to not identify any particular risk model. All numbers in the figure are averages of various variables constructed from monthly portfolios. The computations are based the portfolio managers following one of three factor models, value, momentum, or low beta. C represents our crowding measure as described in the paper, $C = \frac{\sum_{i=1}^{M}\sum_{j=1}^{M} S_{p:i,j} - M}{M^2-M}$. Ω measures the relative crowding between random signals and actual portfolios, $\Omega = \frac{\sum_{i=1}^{M}\sum_{j=1}^{M} S_{p:i,j} - M}{\sum_{i=1}^{M}\sum_{j=1}^{M} S_{\alpha:i,j} - M}$. A higher value means that risk model creates more crowding. $S.R.$ is a pseudo-Sharpe ratio for each portfolio defined as the portfolio's annualized forward one-month return minus transaction costs divided by its ex-ante standard deviation. Max represents the median of the maximum weight of any portfolio over all months, Min represents the median of the minimum weight of any portfolio over all months, and N represents the average number of stocks across portfolios in any given month over all months. The optimizations represent optimizations which attempt to minimize the variance of the portfolio subject to a target alpha subject to various constraints as explained in the paper. Two transaction cost models are considered, model 1 (TC1) and model 2 (TC2). The transaction costs include spreads and market impact of the following form: $tc_{it} = \left|\frac{100 s_{it}/2}{p_{it}}\right| + |c_{it}|$, where s_{it} is the bid-ask spread of stock i at time t. For model 1, $c_{it} = \frac{I}{2} + \text{sgn}(n_{it})\eta\sigma_{it}\left|\frac{n_{it}}{V_{it}T}\right|^{3/5}$, where $I = \gamma\sigma_{it}\frac{n_{it}}{V_{it}}\left(\frac{N_{it}}{V_{it}}\right)^{1/4}$, $\gamma = 0.314$, $\eta = 0.142$, σ_{it} is the daily volatility of stock return i at the beginning of month t, N_{it} is the total amount of shares outstanding in the security, V_{it} is the average daily trading volume of the stock, T is the time interval in which the trade takes place in number of days, for this paper we use $T = 1$, and n_{it} represents the number of shares of the security the portfolio is trading. For model 2, $c_{it} = B_{it}|n_{it}| + C_{it}|n_{it}|^{0.5}$, where B_{it} and C_{it} are parameters estimated by Northfield, n_{it} is the number of shares to be purchased for security i in month t, and c_{it} is expressed in terms of percentage price movement. Model 2 exists only since March 2009. ***, ** indicates the 99 and 95% significant difference respectively in the average crowding from this portfolio and a portfolio that doesn't consider transaction costs. MN is for the market neutral portfolios and LONG is for the long portfolios.

benchmarks (Chinco and Fos 2016), when they follow similar investing strategies, or even as an unintended consequence of using similar methods to build portfolios. One of the ways that portfolios can become very similar is due to the portfolio construction process, like the use of similar transaction cost models to manage the direct drag (spreads) and the indirect drag (market impact) from trading securities. The simulations in this paper provide evidence that as equity portfolios grow in size from \$500 million each to \$5 billion, crowding actually declines. Thus, for portfolio managers managing less than \$5 billion and using similar transaction cost models and reasonable portfolio construction parameters, the unintended crowding from transaction cost models in the equity space may not be a major concern.

One of the explanations for lower crowding due to transaction costs as portfolios grow from \$500 million to \$5 billion is that as market impact costs become larger in a portfolio, it makes trading large amounts of particular stocks incredibly costly due to the non-linear nature of market impact costs. Thus, a portfolio optimizer will find it advantageous, *ceteris paribus*, to trade very small amounts of many more stocks.† Of course, this is absent any particular alpha considerations. This logical behaviour of the optimizer will result in less crowding amongst similarly optimized portfolios up to a certain portfolio size relative to the security universe.‡ However, as the value of portfolios approaches \$20 billion, crowding due to transaction cost parameters starts to increase and should be of concern.

The results of the paper also suggest that portfolio managers can alleviate the severity of the unintended crowding by assuming a larger asset size when constructing portfolios. This can be understood by comparing the \$500 million case with the \$20 billion case. If portfolio managers construct portfolios using a larger asset base than their own portfolio value to estimate transaction costs, this may lead to portfolios constructed that have less crowding than if they only consider the actual size of their own portfolios. Considering a larger asset base for transaction costs in portfolio construction would make sense if portfolio managers recognize that many similar investors to themselves might rebalance or trade at a similar time as them.

Some portfolio managers do not even consider transaction costs in their portfolio construction, which can cause low ex-post returns as well as unintended crowding. Dan deBartolomeo, the CEO of Northfield, has said 'There is a disconnect in the industry between portfolio construction and trading and many portfolio managers leave the issue of transaction costs to the trading team.'§ Portfolio managers who do ignore transaction costs may create 'crowding' due to transaction costs that will only be realized *ex-post* when it is too late.

The main focus of this paper was to study whether crowding could result when equity portfolio managers use the same transaction cost models to construct their portfolios. Whether or not this crowding will result in a distorted risk and return space will depend on other factors not covered in this research. In particular, if the group of portfolio managers that are managing assets are small relative to the universe of securities, then the crowding of this space by these portfolio managers may not be a concern, since they can trade in and out of their positions with ease. On the other hand, if these portfolio managers in the crowded space are large relative to the universe, than their crowding may distort the risk and return space and cause instability. The question of what is small and what is large is not easy to measure either. Small does not necessarily mean the percentage of assets owned in the space. It may be more relevant to measure the size of the managers' assets under management in relation to the daily liquidity or traded volume of the space. The quant crisis of August 2007 showed that even in a market where the quant players were small relative to total ownership, simultaneous trading in similar securities can cause instability. The determination of when crowding is dangerous or not is a very important one and is for further research to answer.

†A portfolio manager might split up the order over several days, but market impact is still present to a degree, since this is just a scaling of the magnitude of the impact. Even if a portfolio manager builds their positions over several days, they are essentially crowding the investment space regardless and may be in jeopardy when a shock arises that requires them to sell quickly.

‡Of course, there is a complicated relationship between the optimization parameters, the alpha signals, and the constraints of the optimization problem.

§This was said in a conversation with me.

6. Conclusion

The links between market participants' interconnectedness and financial stability are gaining increasing attention in the financial community. The behaviour of market participants can lead to changed equilibrium prices that depart from pricing fundamentals and leave a trading space vulnerable to collapse. One way a trading space can become crowded is through portfolio managers copying each other's trade ideas or implementing similar trade ideas that lead to similar positions. The crowding of the investment space may in turn lead to mismeasurements of risk.

Another way a trading space can become crowded and make it difficult for portfolio managers whose positions have become concentrated to trade those positions might come from the use of similar transaction cost models. Transaction costs influence the net alpha of a stock and hence can potentially influence the type of portfolio an investor chooses. Transaction costs are mainly driven by market impact costs which, in turn, depend on the size of the portfolio.¶ As an example, suppose that two portfolio managers wish to trade two stocks; portfolio manager 1 likes stock A and portfolio manager 2 likes stock B. Ignoring diversification issues, portfolio manager 1 would like to buy 70% of A and 30% of B, whilst portfolio manager 2 would like to buy 30% of A and 70% of B. However, if stock B has a large transaction cost relative to A, then both managers might tilt more heavily towards A. In fact, the result might be that portfolio manager 1 buys 75% of A and portfolio manager 2 buys 70% of A, which causes crowding and ironically may lead to *ex-post* trading costs that are even larger than *ex-ante* trading costs.

Using simulated portfolios, we find that consideration of transaction costs in portfolio construction actually leads to less crowding or at most an insignificant amount of additional crowding for individual portfolio sizes of \$500 million to \$5

¶These costs also depend on the size of other portfolios that are selling or buying securities at the same time.

billion. For long-only portfolio managers with randomly generated alpha models, we find that the average crowding from portfolios with no transaction costs is about 23% greater than crowding from portfolios with an average size of $20 billion that consider transaction costs. For market neutral portfolios, we find no statistical difference in the average crowding that occurs with portfolios up to $20 billion in size. Thus, for portfolio sizes up to $20 billion, the negative impact of transaction costs is not great enough to cause a particular concentration of stock holdings.

However, crowding starts to increase as portfolio sizes grow larger than $20 billion in a US stock universe of 2000 companies over the period from 2006 to 2013. As portfolios grow above this size, the non-linear nature of market impact makes it extremely costly to hold smaller companies, and thus the holdings of the portfolios concentrate and result in increased crowding. Ultimately, the conditions that will cause crowding from transaction costs depend on the ratio of the portfolio size to the average volume of individual securities that are traded. Thus, although portfolio managers during the quant crisis of 2007 mentioned transaction costs as a potential cause of the crowding that occurred during the crisis (Chincarini 2012), the simulated evidence in this paper indicates that the crisis may have had less to do with transaction costs and more to do with other factors such as copycat alpha models. In summary, even though transaction costs and crowding are ultimately important for understanding liquidity and systemic risk, we do not find that they lead to fragile investment conditions for reasonably sized equity portfolios.

Our paper contributes to a better understanding of systemic risk by demonstrating how the interactions of portfolio managers when preparing portfolios may lead to inadvertent crowding. With its particular focus on transaction cost models, the paper shows how crowding and transaction costs are related and also introduces a simple and very useful method to incorporate transaction costs into a portfolio optimization framework.

There are many directions for further research in the area of crowding. A more detailed investigation of the trade-off between trading liquidity and portfolio size would be interesting, including an examination of whether there are obvious limits to a portfolio's size given a trading strategy. An investigation of how different parameters of the portfolio construction process influence the relationship between crowding and transaction costs would also be interesting. Growth in the size of a portfolio may also provide a linkage between what constitutes a short-term investor and a longer-term investor, since transaction cost constraints will force an honest and knowledgeable manager to be a longer-term investor.

Several studies have related crowding and past performance of stock returns (momentum). It might be illuminating to study the links between momentum and transaction costs. That is, as certain stocks perform relatively better, they naturally become a larger proportion of one's portfolio. The future sale of these particular stocks might thus have a larger impact on prices, and hence transaction costs, than originally anticipated. A related area is the question of how a portfolio manager's effective universe of securities decreases as the portfolio size grows and how crowding depends on the size of the investment universe and the number of managers in that universe.

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Appendix 1. Applied optimization details

Our optimizations involve three portfolio management techniques. This appendix describes the optimization problem set-up. All of our optimizations were performed in MATLAB using MATLAB's optimization routines, in addition to user-adjusted optimization routines, and the CPLEX optimization tools from IBM.†

A.1. The long portfolio

Our approach is to maximize the expected return after transaction costs (or net alpha signal) of the portfolio subject to a variety of constraints, including that the portfolio volatility be equal to the 60-month historical volatility of the S&P 500‡, the weights of the portfolio sum to 1, the weights of any individual stock are between 0 and 10%, and that the portfolio has the same exposure to each sector as the benchmark universe of 2000 stocks.

$$\max_{\mathbf{w}} \mathbf{w}'\boldsymbol{\mu} - \widetilde{\mathbf{tc}} \tag{A1}$$
$$\text{s.t.} \tag{A2}$$
$$\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} = \sigma_{S\&P500} \tag{A3}$$
$$\mathbf{w}'\boldsymbol{\iota} = 1 \tag{A4}$$
$$0 \leq \mathbf{w} \leq 0.10 \tag{A5}$$
$$\mathbf{S}\mathbf{w} = \mathbf{w}_s^{BM} \tag{A6}$$

where $\mathbf{w}$ are the weights of the stocks in the portfolio, $\boldsymbol{\mu}$ is a vector of alpha signals for each stock, $\widetilde{\mathbf{tc}}$ is the net transaction costs, $\boldsymbol{\Sigma}$ is the variance–covariance matrix of stock returns, $\boldsymbol{\iota}$ is a vector of ones, $\mathbf{S}$ is an M-by-N matrix of zeros and ones representing the M sectors of the economy with a 1 if the security is in that sector and a 0 if not, and $\mathbf{w}_s^{BM}$ is an M-by-1 vector of sector weights for the benchmark universe.

We also consider the reverse optimization problem whereby the portfolio is constructed by minimizing the variance of the portfolio subject to achieving a target after-transaction cost alpha equal to historical annualized volatility of the S&P 500 divided by $\sqrt{12}$.

A.2. The market neutral portfolio

Since many quantitative portfolio managers construct market neutral portfolios, we also investigate crowding with the market neutral construction. The approach is to maximize the expected return after transaction costs or net alpha of the portfolio, whilst constraining the portfolio to have a target volatility equal to 5% over the risk-free rate, have a leverage of 2 and be dollar-neutral (that is, sum of long weights sum to 1 and sum of short weights sum to 1), the long portfolio is sector neutral to the short portfolio, the weights of an individual stock cannot be less than −10% or greater than 10%§, and beta neutral (that is, the weighted average beta of the long portfolio equals the weighted average beta of the short portfolio).

$$\max_{\mathbf{w}} \mathbf{w}'\boldsymbol{\mu} - \widetilde{\mathbf{tc}} \tag{A7}$$
$$\text{s.t.} \tag{A8}$$
$$\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} = 0.05 \tag{A9}$$
$$\mathbf{w}_L'\boldsymbol{\iota} = 1 \quad \forall w_i \geq 0 \tag{A10}$$
$$\mathbf{w}_S'\boldsymbol{\iota} = -1 \quad \forall w_i < 0 \tag{A11}$$
$$-0.10 \leq \mathbf{w} \leq 0.10 \tag{A12}$$
$$\mathbf{w}'\boldsymbol{\beta}|_{w_i \geq 0} = -\mathbf{w}'\boldsymbol{\beta}|_{w_i < 0} \tag{A13}$$
$$\mathbf{S}\mathbf{w}_L = -\mathbf{S}\mathbf{w}_S \tag{A14}$$

where $\mathbf{w}$ are the weights of the stocks in the portfolio, $\boldsymbol{\mu}$ is a vector of alpha signals for each stock, $\boldsymbol{\Sigma}$ is the variance–covariance matrix of stock returns, $\boldsymbol{\iota}$ is a vector of ones, $\boldsymbol{\beta}$ is a vector of the CAPM beta for each stock estimated on five-year historical return data, $\mathbf{S}$ is an M-by-N matrix of zeros and ones representing the M sectors of the economy with a 1 if the security is in that sector and a 0 if not, $\mathbf{w}_L$ and $\mathbf{w}_L$ represents the weights of the long and short portfolio respectively.

We also consider the reverse optimization problem whereby the portfolio is constructed by minimizing the variance of the portfolio subject to achieving a target after-transaction cost alpha equal to historical annualized volatility of the S&P 500 divided by $\sqrt{12}$.

A.3. The market neutral portfolio with liquidity constraints

We constructed market neutral portfolios that incorporated reasonable self-imposed liquidity constraints. The optimization approach was exactly the same as for the market neutral portfolio, however, we added a liquidity purchase constraint that is a fraction of the average daily trading volume.

†Some of the optimizations were not solvable in feasible time with versions of MATLAB older than 2014a.

‡In cases where it is not feasible to achieve the S&P 500 historical volatility, the closest feasible volatility is used in the optimization.

§We initially started with smaller weight restrictions of 0.03 and −0.03, but many of the optimizations could not be solved, thus we expanded the weight constraint.

Liquidity constraints are relatively straightforward to add to the optimization problem. The constraint takes the form of a portfolio manager not wishing to trade more than some percentage of the average daily trading volume of the stock. That is, the constraint is $V_t w_{it} \leq c ADTV_{it}$ or $w_i \leq \frac{c}{V_t} ADTV_{it}$, where c represents a constant indicating the threshold percentage that the portfolio manager wishes to trade in any given stock, V_t is the dollar value of the portfolio, and $ADTV_{it}$ is the average daily trading volume of stock i at time t in dollars. A typical value for this in the quantitative world is 15%.¶

Since the liquidity constraint is essentially an upper bound weight constraint, the upper bound and lower bound weight constraint for every stock was adjusted using the following algorithm. If the liquidity constraint was higher than the existing stock constraint (i.e. 10%), then we didn't alter the stock's weight constraint. If smaller, we changed the upper and lower bound constraint to be equal to the liquidity constraint value for each stock. We did this for both the long and short side of the portfolio.

Unfortunately, when we added these constraints and increased the size of the portfolios, oftentimes there was no feasible solution. Also, since our main goal was to investigate transaction costs and their impact on portfolio construction, we removed the liquidity constraints and did not report them in this paper.

A.4. Market neutral construction

One of the challenges of the market neutral optimization was to set-up the problem so that leverage could be limited. The method we employed for every one of the N stocks in our stock universe was to create an additional set of weights called buy weights and an additional set of sell weights. Thus, for N stocks, we created weights, $w_1 ... w_N$, $w_1^b ... w_N^b$, and $w_1^s ... w_N^s$. We then constructed our entire optimization with these $3N$ weights. In preparing our inputs for the optimization, we formulated the following:

$$\boldsymbol{\mu} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{(A15)}$$

where there are $2N$ zero values in the column. The variance–covariance matrix was also modified as follows:

$$\boldsymbol{\Sigma} = \begin{bmatrix} V(r_1) & C(r_1, r_2) & \cdots & C(r_1, r_N) & 0 & \cdots & 0 \\ C(r_2, r_1) & V(r_2) & \cdots & C(r_2, r_N) & 0 & \cdots & 0 \\ \vdots & & & & 0 & \cdots & 0 \\ C(r_N, r_1) & C(r_N, r_2) & \cdots & V(r_N) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & & & & \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix} \quad \text{(A16)}$$

Most importantly, we altered the constraints in such a way as to keep the main constraints on the final weights (i.e. $w_1 ... w_N$), whilst achieving our market neutral leverage and dollar-neutral constraints. Thus,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & -1 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 0 & 0 \cdots & 0 & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \cdots & 1 & \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \text{(A17)}$$

¶The quantitative manager might also have a total limit on the ultimate size of any position, for example, three times the ADTV. We did not consider this additional consideration for this study.

These constraints created an optimization whereby $w_i = w_i^b - w_i^s$, $\sum_i^{n_b} w_i^b = 1$, and $\sum_i^{n_s} w_i^s = 1$. We also added constraints that $\mathbf{w}_B \geq 0$ and $\mathbf{w}_S \geq 0$, where these are the vector of buy weights and sell weights respectively. This ensured that our market neutral portfolio was dollar neutral and had leverage limited to 2. One could modify this for other forms of leverage very easily. The weights we ultimately are interested in are the $\mathbf{w}$. Any additional constraints on these weights, such as upper and lower bounds or sector constraints were added to the constraint matrix, A, simply by adding rows and placing zeros wherever the $\mathbf{w}_B$ and $\mathbf{w}_S$ occurred.

Although this solution enabled leverage and dollar-neutral constraints on our market neutral portfolio, it did not guarantee that we didn't have wasteful solutions such as purchasing and selling pieces of the same stock. In order to reduce this possibility, we introduced a penalty function into our objective function of the form,

$$-\mathbf{A}(\iota' \mathbf{w_B} + \iota' \mathbf{w_S}).$$

Appendix 2. Transaction costs construction

Due to the recursive nature of transaction costs, that is, the optimal weight of a stock depends on the transaction costs of that stock, but the transaction costs, due to market impact, depends on the optimal weight of the stock, we use the technique outlined in the paper to approximate transaction costs by a quadratic function. In this appendix, we show how to use the results of that approximation to optimize the portfolio.

B.1. Long-only portfolio

In order to incorporate our approximate transaction costs into the portfolio optimization problem, we must modify the quadratic optimization program slightly.† First, we must use a quadratic optimization routine that can accept quadratic constraints, in addition to linear constraints.‡ Second, we must modify the traditional portfolio optimization setup to work with transaction costs.

The mathematical expression of the quadratic optimization with quadratic constraints is given as,

$$\min_{\mathbf{x}} \frac{1}{2} \mathbf{x}' \mathbf{Q} \mathbf{x} + \mathbf{x}' \mathbf{c} \quad s.t. \quad \mathbf{A}' \mathbf{x} \leq \mathbf{b} \quad \text{(B1)}$$

$$\mathbf{l}' \mathbf{x} + \mathbf{x}' \mathbf{Q}^* \mathbf{x} \leq \mathbf{r} \quad \text{(B2)}$$

$$\mathbf{lb} \leq \mathbf{x} \leq \mathbf{ub} \quad \text{(B3)}$$

where $\mathbf{x}$ is the vector of unknowns in the problem, $\mathbf{Q}$ is a symmetric positive semi-definite matrix supplying the coefficients on the quadratic terms of the optimization problem, $\mathbf{c}$ is a vector of coefficients related to the linear objective function, $\mathbf{A}$ is a matrix of coefficients for the equality and inequality constraints, $\mathbf{b}$ is a vector of constraint values, $\mathbf{l}$ is a vector, $\mathbf{Q}^*$ is a matrix, $\mathbf{lb}$ is a lower bound vector, and $\mathbf{ub}$ is an upper bound vector.

In the traditional mean-variance optimization problem, we substitute the following variables; $\mathbf{x} = \mathbf{w}$, the stock weights, $\mathbf{Q} = \boldsymbol{\Sigma}$, the variance–covariance matix of stock returns, $\mathbf{c} = \mathbf{0}$, $\mathbf{l} = \mathbf{0}$, $\mathbf{Q}^* = \mathbf{0}$, $\mathbf{A}$ is chosen typically to have a row of ones and a row of expected returns, and the lower and upper bounds are set as desired.

In order to create an optimal portfolio which minimizes the risk of the portfolio and achieves a desired after-transaction cost alpha, the

†Some books discuss using binary constraints as a way of including transaction costs. Usually, this is because those writers have not considered the empirical implications. It is extremely difficult for the optimizer to solve such problems. In fact, even impossible.

‡For example, CPLEX's cplexqcp.

parameters chosen were as follows:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ [\,] \end{bmatrix} \tag{B4}$$

$$\mathbf{b} = \begin{bmatrix} 1 \\ [\,] \end{bmatrix} \tag{B5}$$

$$\mathbf{Q} = 2\boldsymbol{\Sigma} \tag{B6}$$

$$\mathbf{Q}^* = \begin{bmatrix} \hat{\beta}_1 & 0 & \ldots & 0 \\ 0 & \hat{\beta}_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \hat{\beta}_N \end{bmatrix} \tag{B7}$$

and $\mathbf{l} = -\tilde{\boldsymbol{\mu}}$, $\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} - \hat{\boldsymbol{\alpha}}$, $\mathbf{c} = \mathbf{0}$, and $r = -\mu^T$, where $\tilde{\boldsymbol{\mu}}$ is a vector of the expected returns of each stock minus the constant estimate in the transaction cost regression, $\boldsymbol{\mu}$ is the expected return of each stock, μ^T is the desired target of after transaction costs expected return for the portfolio to match, and $\hat{\beta}_i$ is the coefficient estimate from the transaction cost regression for stock i.

For the dual problem of maximizing the after transaction cost return, whilst achieving a target variance, the parameters chosen are as follows:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ [\,] \end{bmatrix} \tag{B8}$$

$$\mathbf{b} = \begin{bmatrix} 1 \\ [\,] \end{bmatrix} \tag{B9}$$

$$\mathbf{Q} = \begin{bmatrix} \hat{\beta}_1 & 0 & \ldots & 0 \\ 0 & \hat{\beta}_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \hat{\beta}_N \end{bmatrix} \tag{B10}$$

$$\mathbf{Q}^* = \boldsymbol{\Sigma} \tag{B11}$$

and $\mathbf{l} = \mathbf{0}$, $\mathbf{c} = -\tilde{\boldsymbol{\mu}}$, $\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} - \hat{\boldsymbol{\alpha}}$, and $r = \sigma^T$, where $\tilde{\boldsymbol{\mu}}$ is a vector of the expected returns of each stock minus the constant estimate in the transaction cost regression, $\boldsymbol{\mu}$ is the expected return of each stock, σ^T is the target volatility for the portfolio to match, and $\hat{\beta}_i$ is the coefficient estimate from the transaction cost regression for stock i.

B.2. Market neutral

The market neutral problem is slightly more complicated. As explained previously, we create phantom weights for the securities in the long and the short portfolios. In order to create an optimal portfolio which minimizes the risk of the portfolio and achieves a desired after-transaction cost alpha, the parameters chosen were as follows:

$$\mathbf{A} = \begin{bmatrix} [\,] \end{bmatrix} \tag{B12}$$

$$\mathbf{b} = \begin{bmatrix} [\,] \end{bmatrix} \tag{B13}$$

$$\mathbf{Q} = 2\boldsymbol{\Sigma} \tag{B14}$$

where $\boldsymbol{\Sigma}$ is as in equation (A16).

$$\mathbf{Q}^* = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Sigma}_2 \end{bmatrix} \tag{B15}$$

where

$$\boldsymbol{\Sigma}_2 = \begin{bmatrix} \hat{\beta}_1 & 0 & \ldots & 0 \\ 0 & \hat{\beta}_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \hat{\beta}_N \end{bmatrix} \tag{B16}$$

and $\mathbf{l} = -\tilde{\boldsymbol{\mu}}$, $\tilde{\boldsymbol{\mu}} = [\boldsymbol{\mu}, -\hat{\boldsymbol{\alpha}}, -\hat{\boldsymbol{\alpha}}]'$, $\mathbf{c} = \mathbf{0}$, and $r = -\mu^T$, where $\tilde{\boldsymbol{\mu}}$ is a $3 \times N$ matrix of the expected returns of each stock and the constant estimates in the transaction cost regression, $\boldsymbol{\mu}$ is the expected return of each stock, μ^T is the desired target of after transaction costs expected return for the portfolio to match, and $\hat{\beta}_i$ is the coefficient estimate from the transaction cost regression for stock i.

For the dual problem of maximizing the after transaction cost return, whilst achieving a target variance, the parameters chosen are as follows:

$$\mathbf{A} = \begin{bmatrix} [] \end{bmatrix} \tag{B17}$$

$$\mathbf{b} = \begin{bmatrix} [] \end{bmatrix} \tag{B18}$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Sigma}_2 \end{bmatrix} \tag{B19}$$

where

$$\boldsymbol{\Sigma}_2 = \begin{bmatrix} \hat{\beta}_1 & 0 & \ldots & 0 \\ 0 & \hat{\beta}_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \hat{\beta}_N \end{bmatrix} \tag{B20}$$

$$\mathbf{Q}^* = 2\boldsymbol{\Sigma} \tag{B21}$$

where $\boldsymbol{\Sigma}$ is as in equation (A16) and $\mathbf{l} = \mathbf{0}$, $\mathbf{c} = -\tilde{\boldsymbol{\mu}}$, $\tilde{\boldsymbol{\mu}} = [\boldsymbol{\mu}, -\hat{\boldsymbol{\alpha}}, -\hat{\boldsymbol{\alpha}}]'$, and $r = -\sigma^T$, where $\tilde{\boldsymbol{\mu}}$ is a $3 \times N$ matrix of the expected returns of each stock and the constant estimates in the transaction cost regression, $\boldsymbol{\mu}$ is the expected return of each stock, σ^T is the after transaction costs risk for the portfolio to match, and $\hat{\beta}_i$ is the coefficient estimate from the transaction cost regression for stock i.